Confronting Stupidity
and
Ultimately Failing

"Knowledge is power." Humans love this phrase. They repeat it through years of formal education, of work, of hardship, of family-rearing, of life -- all to remind themselves to stay curious, to rebuke stagnation, and to continually strive for greater understanding. To live by this mantra is to lead life with a question, to sate one's curiosity before one's stomach, and to fall asleep wondering rather than reflecting. In practice, however, the quest for knowledge is one of fear, a means for humans to reconcile with their own impermanence. For them, "Ignorance is mortality."

A room is a curious thing. For 23 hours a day, desks sit attentive before a bare board. The air is a thick, dense, motionless courier laced with dust. Aside from an occasional sigh, the room slumbers in perfect silence. So it remains – until 15:05 when the handle turns, the switch is flicked, and a muted assortment of weary students filter through a just-too-narrow doorway. Yellowed walls muffle their polite, pre-talk chatter as they distribute snacks and assume their seats. One by one, backs straighten, feet face front, and eyes lock gaze with a still blank board, which looms ominously far above the back wall.

The Lecturer sits at front; motionless, barring an occasional bob of the leg. Right cheek faces the board, left cheek the students -- but partially closed eyes stare thoughtfully forward and look upon neither. This lone actor straddles the line between blank board and barren brains with chalk in hand, poised to fill both at once. Something will happen in this slightly-too-warm room, but for now, at 15:09, the brief burst of sound decays into silence. One does not create mathematics, it happens, and as the tide recedes before a storm, so does motion hush before logic and art may dance.

Gasp*, *gasp*, did I lock my bike? I reallllly don't think I locked my bike. *Sniff Mark pauses in the hallway, tempted to race back up the stairs. He pulls out his phone – *15:09? I'm already late!* – before awkwardly racing away. His backpack writhes against its straps as he jogs away amidst the fading jingling of erratic zippers.

The room snaps from its stupor as the door swings open and Mark staggers inside, failing to hold back hist loud gasps for air. He shambles down the column of students and elbows his way into the second row, apologizing between still labored breaths.

Zip*. Whoops. Wrong zipper. *Zip*. *Ziiiiiip*.* Mark yanks out a pad of notes and begins riffling through its pages. He pauses for a moment on a single blank page in his otherwise filled notebook. *Screw that random blank page -- I don't know when, but one time I flipped too far and skipped a couple of pages and now I have this pristine sheet of paper uncomfortably embedded between two back-to-back lectures. I would remove it, except for these cheap notebooks never rip along the perforations and always leave those little nubs of paper that you can't pull out without bending the spirals and I HATE those nubs, because you know they're gonna stay there the* *whole semester, and then the pages won't turn correctly.* The Lecturer suddenly stands and turns to the board. *Agh, the Lecturer is starting to write! Where's my pencil, did I pull it out already? *ZIP* Oh, it's on the floor. *ZIP* "psst hey, sorry, my pencil fell under your chair...thank you so much...sorry..." Oh dear: it's out of lead. *ZIP

Mark fumbles with his pencil as the Lecturer begins to speak. "Welcome. Today, we will discuss a niche topic in dimension theory: arithmetic rank," the Lecturer says. "Throughout this lecture, '*R*' is a commutative Noetherian ring, 'dim' is Krull dimension, and 'height' is the height of an ideal." Mark looks up at the Lecturer, and nods along with a few other scattered students in silent affirmation. "Recall from our last lecture that if two ideals, '$\mathfrak{a}$' and '$\mathfrak{b}$', have the same radical, then they define the same local cohomologies. This suggests the following definition." The Lecturer pauses and writes on the board:

Definition 1. Let $\mathfrak{a}$ be an ideal in R. The arithmetic rank of $\mathfrak{a}$ is the number

$$\operatorname{ara}\mathfrak{a} = \inf\left\{ \nu(\mathfrak{b}) \mid \mathfrak{b} \text{ is an ideal with } \sqrt{\mathfrak{b}} = \sqrt{\mathfrak{a}} \right\}$$

where $\nu(\mathfrak{b})$ is the minimal number of generators of the ideal $\mathfrak{b}$.

**Flip* *flip* *flip* How did we define local-cohomology again? It's in here somewhere...here it is. We talked about this two weeks ago, I should really remember it by now.*

The Lecturer continues, "Consider for a moment the connection between this definition and the Nullstellensatz. Here, the arithmetic rank of an ideal can be thought of as the minimal number of equations needed to define the vanishing set associated to the ideal – it is the minimal number of hyper-surfaces needed to cut out its algebraic set in affine space."

Mark glances up from his notebook. *Hyper-surface in affine space? What the hell is that? I'd better check the textbook – wouldn't want to write down the wrong thing.*

"As you might expect, arithmetic rank fits nicely with other invariants we have encountered. Consider the following proposition." Mark furrows his brow as the Lecturer writes,

Proposition 1. Let R be a Noetherian ring and $\mathfrak{a}$ an ideal of height h. Then $H^h_{\mathfrak{a}}(R)$ is nonzero and

$$\operatorname{height}\mathfrak{a} \leq \operatorname{cd}_R(\mathfrak{a}) \leq \operatorname{ara}\mathfrak{a}.$$

"And here is the proof," the Lecturer says.

Once again, Mark looks up from the pages of his textbook in despair. *We're already proving something? I still haven't copied down that first definition...wait no!*

Mark's hand freezes in defeat as the Lecturer sweeps an eraser across the board. Letters and symbols smear into uniform streaks under the entropic pad. In thick print, the Lecturer writes, "*Proof:*"

I hadn't written that down yet! I'm so slow -- I really should have slept last night. It's alright, I can still get that proposition. What's that funny looking 'a' mean again?

"For any ideal '$\mathfrak{b}$' with '$\sqrt{\mathfrak{b}} = \sqrt{\mathfrak{a}}$,' we have $\mathrm{cd}_R(\mathfrak{a}) = \mathrm{cd}_R(\mathfrak{b})$ by Proposition 7.3.2," says the Lecturer. "As you all know from our previous lectures, '$\mathrm{cd}_R(\mathfrak{b}) \leq \nu(\mathfrak{b})$' by Corollary 7.14. This clearly proves the inequality on the right..."

Mark scribbles frantically. *I'm already so far behind, I'll have to look up those corollaries and propositions later. We covered those in a previous lecture? I haven't missed a single one, why can't I remember them?!*

"...Now, let '$\mathfrak{p}$' be a prime containing $\mathfrak{a}$ with $\mathrm{height}(\mathfrak{p}) = h$." The Lecturer turns, and asks, "Any questions?" Mark opens his mouth but the Lecturer immediately continues, "We then have the following chain of isomorphisms." The Lecturer falls silent, and writes,

$$\boxed{H^h_{\mathfrak{a}}(R)_{\mathfrak{p}} \cong H^h_{\mathfrak{a}R_{\mathfrak{p}}}\left(R_{\mathfrak{p}}\right) \cong H^h_{\mathfrak{p}R_{\mathfrak{p}}}\left(R_{\mathfrak{p}}\right).}$$

Mark's heart beats faster. *What on EARTH is that?? Does that subscript have a subscript? I don't know what any of this means! Slow down, please for the love of God slow down!*

"The first isomorphism is given by Proposition 7.15.3, and the second by Proposition 7.3.2. But $\mathrm{H}^h_{\mathfrak{p}R_{\mathfrak{p}}}\left(R_{\mathfrak{p}}\right) \neq 0$ by Theorem 9.3, so the above isomorphisms imply that $\mathrm{H}^h_{\mathfrak{a}}(R) \neq 0$, and that settles the inequality on the left. And that's the proof."

The Lecturer finishes, and without turning, asks, "Any questions?" It's more a challenge, than anything else. Mark wears a practiced, stoic face, mastered over many seminars and class periods, his questions lost to confusion. The untrained eye would glance over his neutral, slightly quizzical expression without noticing his clenched jaw or white knuckles.

...Please, someone ask a question...give me more time. I need the rest of that proof; how can I possibly understand it if I can't even copy it down correctly? Algebra is supposed to be my forte, my specialty...what the hell *is a hyper-surface in affine space? Is the Lecturer* conjugating *a ring by a prime element in that third isomorphism? Does that question even make sense? I really wish I had slept last night...*

Chalk is clacking again; if only for this one hour, the board demands to be written upon. "This seemingly trivial proposition" – *trivial??* – "leads to an important result regarding the arithmetic rank of ideals in local rings. In its original form, this was due to Kronecker, who formulated this in his final year of undergraduate study." The Lecturer's arm rises, eraser in hand, and in a flash the board is again black. The ghost of a proof fades in a faint cloud of eraser dust.

You need to calm down, this is only one seminar. You can copy down notes from the textbook later...after you finish the notes from last week. And the week before that. Have you ever successfully listened to an entire lecture? No wonder you don't understand! You won't even admit how far behind you've fallen!

It is a little-known fact that too many late nights, too many cups of coffee, and too many small failures can cause acute hearing loss. The Lecturer continues, but words fall flat against Mark's ears. To him, the world is silent. All that remains is the board. His eyes remain transfixed, and his hand scribbles mindlessly, copying the theorem emerging in front of him.

Theorem 1. *If a chain complex C of free Abelian groups has homology $H_n(C)$ and if the direct limit of the system C_n/C_{n+1} comes equipped with a presheaf $\mathcal{F}$, then the cohomology groups $H^n_{\mathcal{F}^*}(C;G)$ of the cochain complex* $\mathrm{Hom}_R(C_n, G)$ *are determined by split exact sequences*

$$0 \xrightarrow{\phi} \mathrm{Ext}(H_{n-1}(C), G) \xrightarrow{\psi} H^n(C;G) \xrightarrow{h} \mathrm{Hom}_R(H_n(C), G) \to 0,$$

and $\mathrm{ara}\ \mathfrak{a} \leq \dim R$.

Chalk snaps, the Lecturer pivots. Mark's practiced stoicism betrays him as a wave of fear washes across his face and he locks eyes with the Lecturer. "Is everything alright?" asks the Lecturer.

...No. No it is not...please continue...

Met with naught but silence, the Lecturer glances at the board and then back to Mark. "You took my algebra class. Tell me, why are we guaranteed that this sequence is exact?" Mark's mouth hangs agape, yet he is unable to speak. The Lecturer points at the series of arrows on the board, "This particular short exact sequence is induced by the long exact sequence on our chain complex C." The Lecturer now points at Hom, “Hom is always a left-exact functor, but here it is also right-exact. Why is that?"

I don't know, I don't know...please just tell me and move on with the lecture. Pleeaase just move on...

But the Lecturer isn't finished. "You've taken four of my classes, Mark, and yet you can't recall the basic properties of Hom? I wrote a letter of recommendation for you last week; do you mean to tell me you've been wasting my time? What about $\mathrm{H}_n(C)$, can you at least define *that* for me? The homology of C, *what is it*?!" Silence throbs in the air, pressing down on Mark. "Unbelievable! A letter for and undergraduate conference at MSRI, you ask? MSRI?! You couldn't list the words in the acronym! In all this time, Mark, have you truly learned nothing? What is a hyper-surface in affine space? Surely, SURELY you can define at least that."

Mark is visibly trembling now; any pretense of calm has evaporated. Gone is comprehension and reason, and here is the unshakeable, omnipotent sense of stupidity, enveloping him, clouding his judgement, overwhelming his sense of self. His own inadequacy beats against the inside of his skull, an entity fighting for total control. *I'm so stupid, I'm so stupid, I'm so stupid, I've never heard of a hyper-surface before, why am I SO STUPID...*

"Do you not think I notice when you come up short," the Lecturer spits, "when you stare at a problem for hours, without a glimmer of a notion of where to start, when you exit class in a daze, having understood nothing, when you sit down at an exam and freeze on the first page? Ignorance is your default state, stupidity is the banner under which you march! Who are you, to even DREAM of studying math, to walk the halls of knowledge? You aren't worthy!"

I am not worthy, I am stupid.

"You seem to think that there is some reason you should sit here, week after week. You ought to have walked away years ago, but where would you go? Where would you turn? What is your purpose, Mark? *Why are you here*?"

The Lecturer is right: math is more than the source of meaning in my life – it's where I've piled all my chips. If my career doesn't pan out, then I will have wasted each and every one of my 20 years on this earth. The entirety of my purpose hinges on this success, and if I continue to fail, then for what reason do I remain alive?

A room is a curious thing. For 23 hours a day, walls stand still and time ticks steadily onward. However, at 15:33 – or is it 15:49? – the edges of the board stretch off past the horizon. Strokes of chalk become glints of light scattered across an infinite expanse of black. The far wall warps and twists until it is a distant street winding two hundred feet below the classroom, now exposed to the open air. The room, once too warm, is suddenly chilled, yet still there is no breeze. Students disappear, stucco vanishes, and the Lecturer takes his place among the stars. Mark, alone, stands perched on the ledge – or is it his seat? – suspended between earth and sky. He stands on the wrong side of the railing, knuckles still white, fingers clenched around cold steel, staring down at the pavement below.

I should jump.

In a way, the pursuit of knowledge is thankless. Hawking said of death and god, "I regard the brain as a computer which will stop working when its components fail. There is no heaven or afterlife for dead computers; that is a fairy story for people afraid of the dark." Stephen Hawking donated his life to the pursuit of knowledge, yet the only thanks afforded a physicist is the dark abyss of space, and a cold, permanent sleep.

I can imagine myself falling, soaring through the air and into the ground below. The sudden weightlessness, the rush of air, it's all so familiar, and yet I hesitate...have I been here before...?

The true irony of Hawking's elevated sense of purpose is that the ultimate fruits of his labor are, almost tautologically, unattainable. Hawking would have it no other way. When the final scrap of knowledge is catalogued, the final theorem proves, humanity will have completed its only enduring goal. If the purpose of life is to expand the bounds of knowledge – to learn – then what is left once all is discovered and all is known? Mark lifts his gaze from the street, and stares deep into the sky.

...when I failed to submit my Princeton application, I came here. When I failed my first physics exam, I came here. When I read late into the night without understanding, when I quit the lab in despair, when I forgot my lines in the middle of a talk – I came here. I will again fall to ignorance and will again return to this ledge distraught. Indeed, stupidity is my default state, but to truly overcome it is not only impossible, it is undesirable. I, like Hawking, will never see the Theory of Everything scrawled across the blackboard. To witness such an event would be to witness the ultimate conclusion, for to a scientist, the Theory of Everything is indistinguishable from the End of Everything. I seek after knowledge, yet accept I will forever fail in my pursuit.

Mark drops his gaze and lifts himself over the railing. He steps off the ledge, back onto solid ground, and walks away slowly. His zippers cease their jingling for only an instant, when he pauses, flicks the switch, and shuts the door. The time is 16:05, and the now-empty room returns to slumber.

There is a wide, yawning black infinity. In every direction, the extension is endless; the sensation of depth is overwhelming. And the darkness is immortal. Where light exists, it is pure, blazing, fierce; but light exists almost nowhere, and the blackness itself is also pure and blazing and fierce. But most of all, there is very nearly nothing in the dark; except for little bits here and there, often associated with the light, this infinite receptacle is empty. This picture is strangely frightening. It should be familiar. It is our universe.

– Carl Sagan