Physics 550

Problem Set 6: Kinematics and Dynamics

Name: ______________________________

Instructions / Notes / Suggestions:

- Each problem is worth five points.
- In order to receive credit, you must show your work.
- Circle your final answer.
- Unless otherwise specified, answers should be given in terms of the variables in the problem and/or physical constants and/or cartesian unit vectors ($\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_x, \hat{\mathbf{e}}_z$) or radial unit vector $\hat{\mathbf{r}}$.
- Staple your work.

Problem #1:

The Kronecker product of two matrices:

$$\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$$

is defined in terms of their matrix elements:

$$C_{(ik)(jl)} = A_{ij} B_{kl}$$

Show that:

$$\mathrm{Tr}\left(\mathbf{C}\right) = \mathrm{Tr}\left(\mathbf{A}\right) \mathrm{Tr}\left(\mathbf{B}\right)$$

Problem #2:

The Hamiltonian H of a two-component, non-interacting system may be written:

$$H = H_1 \otimes I + I \otimes H_2$$

where H_1 and H_2 are the Hamiltonians of the two components, and I is the identity operator.

Show that the corresponding time evolution operator $U(t)$ has the form:

$$U(t) = U_1(t) \otimes U_2(t)$$

where:

$$\begin{aligned} U_1(t) &= \exp(-itH_1/\hbar) \\ U_2(t) &= \exp(-itH_2/\hbar) \end{aligned}$$

Problem #3:

Show that a pure state cannot evolve into a non-pure state, and vice versa.

Problem #4:

Consider an arbitrary physical system.

Denote the Hamiltonian of this system by $H_0(t)$, and the corresponding evolution operator by $U_0(t, t_0)$:

$$i\hbar \frac{\partial}{\partial t} U(t, t_0) = H_0(t) U_0(t, t_0)$$
$$U_0(t_0, t_0) = I$$

Now suppose that the system is perturbed in such a way that its Hamiltonian becomes:

$$H(t) = H_0(t) + H_1(t)$$

The so-called "interaction picture" (I) may be obtained by the following transformation of states and dynamical variables of the Schrödinger picture (S):

$$|\psi_I(t)\rangle = U_0^\dagger(t, t_0) \, |\psi_S(t)\rangle$$
$$\rho_I(t) = U_0^\dagger(t, t_0) \rho_S(t) U_0(t, t_0)$$
$$R_I(t) = U_0^\dagger(t, t_0) R_S U_0(t, t_0)$$

(a) Find the equations of motion for the state vector $|\psi_I(t)\rangle$ and state operator $\rho_I(t)$.
(b) Explain why (qualitatively), when $H_1(t)$ is much smaller than $H_0(t)$, the motion of $|\psi_I(t)\rangle$ is much slower than that of $|\psi_S(t)\rangle$.
(c) Determine the time dependence of the average of the observable represented by the operator R. Compare this to that in the Schrödinger and Heisenberg pictures.

Problem #5:

The unitary operator:

$$U(\mathbf{v}) = \exp(i\mathbf{v} \cdot \mathbf{G})$$

describes the instantaneous ($t = 0$) effect of a transformation to a frame of reference moving at the velocity $\mathbf{v}$ with respect to the original reference frame.

The effect of $U(\mathbf{v})$ on the position $\mathbf{q}$ and momentum $\mathbf{p}$ operators are:

$$U\mathbf{q}U^{-1} = \mathbf{q}$$
$$U\mathbf{p}U^{-1} = m\left(\mathbf{v} - \mathbf{v}I\right)$$

where m is the mass (of the quantum state).

Find an operator $\mathbf{G}(t)$ such that the unitary operator:

$$U(\mathbf{v}, t) = \exp(i\mathbf{v} \cdot \mathbf{G}(t))$$

will yield the full Galilei transformation:

$$U\mathbf{q}U^{-1} = \mathbf{q} - t\mathbf{v}I$$
$$U\mathbf{p}U^{-1} = \mathbf{p} - m\mathbf{v}I$$