

# Spectral fluctuations and transport in phase space

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**Abstract.** We extend the semiclassical theory of spectral fluctuations to include composite systems. These systems are characterized by some number of weakly communicating, highly chaotic subsystems. They contain new and longer time scales, depending on the connecting fluxes, in addition to those associated with the rapid internal mixings. The new time scales produce discernible modifications in the nature of the level fluctuations of the corresponding quantum systems. The theory is applied to two coupled quartic oscillators.

## 1. Introduction

The asymptotic properties of simple quantum mechanical systems reflect in various ways the nature of their underlying classical counterparts. In recent years, intriguing relationships between spectral fluctuation properties and chaotic classical dynamics have been discovered. The relationships encountered depend strongly on whether the system of interest is bounded as exemplified by the stadium billiard or whether it is open such as the kicked rotor. In the former case, it has been conjectured [1] that for ‘sufficiently chaotic’ systems, the quantal spectra are extremely rigid, displaying strong level correlations as found in the canonical ensembles of traditional random matrix theory [2]¶. Much numerical work and the beginnings of a semiclassical theory [4, 5] support this conjecture. The basic tenet underlying the notion of sufficiently chaotic is that the phase-space energy surface gets rapidly mixed everywhere. In a statistical sense, the phase space is featureless and characterized by a single mixing time parameter  $t^*$  which is inversely related to the Kolmogorov entropy (Lyapunov exponent). The similarities to the featureless nature of the canonical random matrix ensembles are not accidental. On the other hand, the rapidly mixing tenet cannot be applied to open systems. For the quantized maps of time-dependent systems, it was first of paramount importance to establish under what circumstances the quasi-energy spectrum would be discrete and the eigenvectors dynamically localized [6], and second to connect the localization length with the classical diffusion rate [7]. From

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¶ This is in stark contrast with Poissonian statistics (absence of correlations) expected in integrable systems [3].

this point of view it is not very surprising that the quasi-energy spectra of dynamically localized systems do not correspond to any of the canonical random matrix ensembles [8, 9]. With these important distinctions, the understanding of fluctuations in open and bounded systems has largely developed independently. Recently though, some progress toward bridging these fields has been made by Dittrich and Smilansky [10]. By considering bounded systems displaying dynamical localization, they have studied the transition between having the localization length very small compared with a system size parameter all the way to the inverse situation. In this paper, we shall add to this progress from a very different tack. We shall not assume the existence of a classical diffusion scale, but rather start from a 'microscopic' picture which has in one limit the diffusion image and in another uniform rapid mixing.

Clearly, requiring a bounded system to be sufficiently chaotic is extremely restrictive and it excludes many systems of physical interest. It is therefore of interest to relax this constraint to move toward an understanding of a larger class of systems. We shall do this by investigating semiclassically the spectral fluctuations of chaotic systems with embedded, isolated partial barriers in their phase spaces. These systems may be thought of as composite systems composed of weakly-coupled, highly chaotic subsystems. It is easy to conjure up examples; for instance, systems of  $2, 3, 4, \dots, N$  chaotic billiards connected by narrow ducts (windows) fit into this scheme. The phase space can no longer be characterized statistically by just the mixing time  $t^*$  and in section 2 we shall show, with a slight generalization of the periodic orbit techniques developed in [4], how the extra structure is manifest in the two-point spectral correlations. A crucial role will be played by the phase space fluxes crossing the barriers. We shall then make connections with the diffusion dominated results of Dittrich and Smilansky [10] in the large- $N$  limit. Imperfect transport barriers in phase space may also arise by dynamical constraints rather than configuration space geometry. In systems with two degrees of freedom, both cantori and broken separatrices are known to provide effective partial isolation [11]. In section 3, we shall apply the generalized theory to a system of two coupled quartic oscillators that were recently treated in detail by Bohigas, Tomsovic and Ullmo (BTU) [12]. One of the particular cases (coupling strength) they treated provides us with a good test of the generalized theory because the phase space is dynamically structured by four partial barriers that are neither too restrictive nor too open and with just five component subsystems, one is neither in the sufficiently chaotic domain nor the diffusive regime. The results presented here will complement the picture of BTU who found the fluctuations to be well described by the properties of generalized random matrix ensembles constructed by purely semiclassical considerations.

## 2. Semiclassical theory of spectral fluctuations

### 2.1. The spectral measures

The relationships we are seeking between classical dynamics and spectral fluctuations are well expressed in a set of so-called two-point correlation functions or measures. These measures give us a quantitative means of characterizing the most important aspects of the fluctuations. To a large extent, all these measures carry the same information and depend only upon two points in a spectrum [2]; see [13] for the original work on the measures.

To understand their origin, consider the level density,  $\rho(s)$ , and its integrated version (the staircase),  $N(s_1, s_2)$ , which are expressed as

$$\begin{aligned}\rho(s) &= \sum_j \delta(s - s_j) \\ N(s_1, s_2) &= \int_{s_1}^{s_2} ds \rho(s).\end{aligned}\tag{2.1}$$

The  $s_j$  are the eigenvalues of the system of interest. For convenience, we are using a scaled energy variable  $s$  that is related to the actual energy  $E$  by  $s = (E - E_0)\langle\rho\rangle_E$  where  $E_0$  is an arbitrary origin and the brackets indicate a local energy averaging. Then the mean level spacing  $D(s) = 1$ . The basic quantity common to all the two-point measures is the probability density of finding distinct levels at  $s_1$  and  $s_2$ . In terms of the level density,  $\rho(s)$ , this two-point density,  $R_2(s_1, s_2)$ , is

$$R_2(s_1, s_2) = \rho(s_1)\rho(s_2) - \delta(s_1 - s_2)\rho(s_1)\tag{2.2}$$

where the  $\delta$ -function term eliminates the self-correlations. In random matrix theory, the measures are defined as averages over an ensemble of matrices with elements selected Gaussian randomly. Then an ergodic theorem [14] tells us that if the measures are calculated spectrally instead, almost all the individual members of the ensemble give the same results as the ensemble. Here with only the spectrum of a single system, the measures are defined as averages over some local region in the spectrum, narrow in energy range, but encompassing many levels. The average behaviour of  $R_2(s_1, s_2)$  then depends on the coordinate,  $r = s_1 - s_2$  and the local mean energy. For convenience, the local mean energy variable can be dropped from the notation ( $R_2(s_1, s_2) \rightarrow R_2(r)$ ,  $N(s_1, s_2) \rightarrow N(r)$ , etc). Note, however, the fluctuations typically do depend on the region of the spectrum being used for local averaging since some dependence will be due to effects coming from the changing underlying classical dynamics with energy.

One of the best choices for a measure is the variance of the number of levels in a spectral window of width  $r$ . Denoted as  $\Sigma^2(r)$ , it is simply given by

$$\begin{aligned}\Sigma^2(r) &= \langle N(r)^2 \rangle - \langle N(r) \rangle^2 \\ &= r - r^2 + 2 \int_0^r ds (r - s) R_2(s)\end{aligned}\tag{2.3}$$

where the brackets are put in the first line only to underscore the local energy averaging aspect. When considered as a function of  $r$ ,  $\Sigma^2(r)$  allows one to infer a great deal from a spectrum including the extent of both level repulsion and long-range spectral rigidity.

Dyson and Mehta dealt extensively with an alternative representation of the two-point density. They defined the two-level form factor,  $b(\tau)$ , which is the Fourier transform [13]

$$b(\tau) = \int_{-\infty}^{\infty} ds e^{2\pi i \tau s} [1 - R_2(s)]\tag{2.4}$$

$\tau = t/2\pi\hbar\langle\rho\rangle_E$  is the natural time scale for the form factor in this problem. This time is such that two eigenvalues one mean level spacing apart give rise to a phase

difference of  $2\pi\tau$  in the quantum propagator ( $\tau = 1$  is the 'break time'). An important property of  $b(\tau)$  is that its large- $\tau$  limit is independent of the fluctuations; for  $\tau \rightarrow \infty$ ,  $b(\tau) \rightarrow 0$ . This depends only on the spectrum being real and discrete, and there are very few special exceptions (such as the harmonic oscillator). The fluctuation properties are determined by  $b(\tau)$ 's small- $\tau$  behaviour and approach to zero. To give two examples, a Poissonian spectrum with its large fluctuations has  $b(\tau) = 0$  for all  $\tau$  while the rigid random matrix spectra have  $b(0) = 1$  and it decreases linearly with small  $\tau$ , approaching 0 for  $\tau > 1$ . The number variance expressed in terms of the form factor becomes [13]

$$\Sigma^2(r) = \int_{-\infty}^{\infty} d\tau \frac{\sin^2(\pi r \tau)}{(\pi \tau)^2} [1 - b(\tau)]. \quad (2.5)$$

Excluding  $\tau = 0$ , the quantity  $1 - b(\tau)$  has a natural dynamical interpretation as being proportional to the mean absolute square of a partial trace of the propagator. The partial trace is over the vector space spanned by the portion of the spectrum being used in the local energy averaging. This interpretation supposes that the spectral range is narrow enough that the level density is effectively constant. Equation (2.5) is the most useful form of  $\Sigma^2(r)$  for the present purposes.

## 2.2. Periodic orbit theory

We outline how periodic orbit theory can be used to calculate fluctuations by extending Berry's theory of spectral rigidity [4]. The essential point is to generalize the sum-rule of Hannay and Ozorio de Almeida [15] and thus provide a semiclassical expression for  $1 - b(\tau)$ . To begin with, the exact density of states,  $\rho(E)$ , is decomposed into its average and fluctuating parts. As is well known, the leading average behaviour is related to the phase space energy surface volume as

$$\langle \rho \rangle_E = \frac{1}{(2\pi\hbar)^d} \int d\mathbf{r} \delta(H - E) \quad (2.6)$$

where  $\mathbf{r} = (\mathbf{q}, \mathbf{p})$  is a point in a  $2d$ -dimensional phase space. In this way, a semiclassical argument fixes the relationship between the physical quantities ( $t, E$ ) and the relevant scaled variables ( $\tau, s$ ). Just as in the preceding section, it will be convenient to continue using  $(\tau, s)$  for treating the fluctuations. The oscillating part of the level density,  $\tilde{\rho}(s)$ , about its locally smoothed average can be expressed in terms of the semiclassical (Gutzwiller) trace formula [16] as

$$\begin{aligned} \tilde{\rho}(s) &= \rho(s) - 1 \\ &\approx \sum_j A_j \exp((S_j + \mu_j)/\hbar) \end{aligned} \quad (2.7)$$

where the sum goes over all the periodic orbits  $j$  with energy  $s$  and, positive and negative repetitions of 'primitive' orbits are considered distinct orbits.  $S_j$  and  $\mu_j$  represent the action along the orbit and the phase associated with the Maslov index respectively.  $A_j$  is expressible in terms of the period  $\tau_j$  and the monodromy matrix,  $M_j$ ,

$$A_j = \frac{\tau_j}{\sqrt{\text{Det}(M_j - I)}}. \quad (2.8)$$

From this starting point, Berry found the periodic orbit expression for the form factor to be

$$1 - b(\tau) \approx \left\langle \sum_{j,k} A_j A_k e^{i(S_j - S_k)/\hbar} \delta(\tau - (\tau_j + \tau_k)/2) \right\rangle_s \quad (2.9)$$

with  $k$  restricted to positive traversals. He argued that for  $\tau \ll 1$ , the off-diagonal terms in the sum vanished on local averaging and

$$\begin{aligned} 1 - b(\tau) &\approx \left\langle \sum_j A_j^2 \delta(\tau - \tau_j) \right\rangle_s \\ &\approx 2\tau I(\tau). \end{aligned} \quad (2.10)$$

The factor 2 in the last form arises from the coherence effect of an orbit with its time-reversed partner (assuming time reversal invariance). The sum  $I(\tau)$ , implicitly defined in the second expression above, can be given a representation as an integral. The form is [15]

$$I(\tau) = \Omega(E) \int d\mathbf{r}_0 \delta^{(2d-1)}(\mathbf{r}_0 - \mathbf{r}_\tau) \delta(H(\mathbf{r}_0) - E) \quad (2.11)$$

where  $\mathbf{r}_\tau$  is the point to which the dynamics takes  $\mathbf{r}_0$  after a time  $\tau$ .  $\delta^{(2d-1)}(\mathbf{r})$  relates to the  $2d - 1$  projections of  $\mathbf{r}$  on the energy shell. The volume of the energy shell at  $E$  is denoted by  $\Omega(E)$ . The classical interpretation of this expression is that it gives the probability of the ratio of returning to the initial point after the time  $\tau$  and the system being found anywhere in phase space ( $1/\Omega(E)$ ). This probabilistic interpretation of  $I(\tau)$  is the central point of the present paper, since now we can apply statistical concepts to evaluate  $I(\tau)$  as long as the time scales involved are longer than the mixing time  $t^*$ , and some phase-space coarse graining is allowed. It is interesting to note that while the quantity  $I(\tau)$  is a classical object, the factor  $\tau$  in equation (2.10) has a quantum origin—it is due to the coherent contributions to the amplitude to stay of points along the periodic orbits. The quantum origin of the factor 2 was explained above. (A detailed discussion of the derivation of equation (2.10) and its applications to a class of solid state problems can be found in [17].)

As a first application, consider a bounded, highly chaotic, and uniformly mixing system. In this case, ergodicity implies that the probability to perform periodic motion for any time larger than the mixing time is uniformly distributed and hence it is  $1/\Omega(E)$ . Therefore  $I(\tau) = 1$  which reproduces the sum rule [15]. Note that the mixing time  $\tau^*$  (the scaled version of the  $t^*$ ) cannot be smaller than the shortest periodic orbit's period. This explains why Berry used the period of the shortest orbit as an estimate for  $\tau^*$  ( $\tau^*$  may sometimes require a better estimation). Interestingly, only two parameters of the chaotic dynamics were used: (i) the energy surface volume that fixes the relation between the physical and the scaled variables; and (ii) the mixing time  $\tau^*$ . It can be shown that the non-ergodic dynamics on short time scales, correspond to non-universal correlations on energy scales which are inversely proportional to  $\tau^*$  (saturation). We mention this for the sake of completeness and will deal no more with this aspect [4, 18].

Writing 1 for  $I(\tau)$  in (2.10) we find that for bounded and uniformly mixing systems we get

$$\begin{aligned} 1 - b(\tau) &= 2\tau & \tau^* < \tau < \frac{1}{2} \\ &= 1 & \tau > \frac{1}{2} \end{aligned} \quad (2.12)$$

which is consistent with the expression derived for the Gaussian orthogonal ensemble (GOE) by random matrix theory for  $\tau$  either very small (yet with  $\tau > \tau^*$ ) or very large. The behaviour at intermediate times has not been derived by semiclassical means and is determined here by continuing with the small  $\tau$  result up to the point where  $b(\tau)$  first reaches zero. By this time the semiclassical arguments summarized here must fail. Since  $b(\tau) \rightarrow 0$  for large  $\tau$ , we insert the limiting value as a rough way of interpolating the two regimes.

### 2.3. Composite systems

The key to treating composite systems is to generalize the statistical arguments for obtaining the behaviour of  $I(\tau)$ . Consider a system which can be decomposed into  $N$  weakly coupled domains in phase space. (One can visualize such a system by considering  $N$  stadia connected by small windows.) Denote the phase-space volumes of the individual domains by  $V_j$ , ( $j = 1, \dots, N$ ) and the total volume  $V = \sum V_j$ . The flux  $\gamma_{jk}$  crossing from the  $j$ th to the  $k$ th domain determines the coupling strengths of the composite system, i.e. the smaller the flux, the weaker the connection. In the stadia example  $\gamma_{jk}$  is the product of a window's cross section by the mean normal velocity (using the scaled time  $\tau$ ) of crossing trajectories. The transition probability per unit time to go from the  $j$ th to the  $k$ th domain is  $\gamma_{jk}/V_j$ .  $I(\tau)$  can be decomposed into  $N$  contributing components,

$$I(\tau) = \sum I_j(\tau). \quad (2.13)$$

The  $I_j(\tau)$  are defined by breaking the integral in equation (2.11) into a sum of integrals each taken only over one phase-space domain,  $V_j$ . Using the probabilistic approach developed earlier, the  $I_j$  can be interpreted as the probabilities of starting and finishing somewhere in the  $j$ th region after propagating a time  $\tau$ . If one assumes that the mixing within each domain takes place on a much shorter time scale than the mixing amongst the various stadia, one can write a set of linear coupled equations which govern the behaviour of the  $I_j$  (master equations). Consider the vector  $\Phi(\tau)$  whose elements are the probabilities of being found in one of the  $V_i$  after a time  $\tau$  subject to the initial condition  $\Phi(0)$ . By choosing the initial condition  $\Phi_j(0) = 1$  and all the other components 0, the probability to remain (or staying probability),  $I_j$ , is equal to the overlap  $(\Phi(0), \Phi(\tau))$ . The master equation for  $\Phi(\tau)$  is

$$-\frac{\partial \Phi(\tau)}{\partial \tau} = M \Phi(\tau) \quad \Phi(\tau) = \begin{pmatrix} \Phi_1(\tau) \\ \Phi_2(\tau) \\ \vdots \\ \Phi_N(\tau) \end{pmatrix} \quad (2.14)$$

where  $M$  is a (Markov) matrix whose elements are given by the flux-to-volume ratios

$$\begin{aligned} M_{jk} &= -\gamma_{jk}/V_j & j \neq k \\ M_{jj} &= \sum_{k \neq j} \gamma_{jk}/V_j. \end{aligned} \quad (2.15)$$

The formal solution of this equation is

$$\Phi(\tau) = e^{-M\tau} \Phi(0). \quad (2.16)$$

The eigenvalues  $\Gamma_j$  of the matrix  $M$  have the following properties: one of the eigenvalues, say  $\Gamma_N$ , always vanishes. This is due to the fact that the probabilities must always add up to unity. We are assuming that there is a path of communication between all the domains, i.e. that  $M$  is not reducible. Otherwise, there would be additional vanishing eigenvalues, one for each subgroup of linked domains. The other eigenvalues may be complex, in which case they appear in conjugate pairs with a positive real part. The  $I_j$  are the diagonal elements of  $e^{-M\tau}$  and so the sum appearing in equation (2.13) is the trace

$$\sum I_j(\tau) = \text{Tr}(e^{-M\tau}) = 1 + \sum_{j=1}^{N-1} e^{-\Gamma_j \tau}. \quad (2.17)$$

The last form follows because the diagonalizing transformation of  $M$  leaves the trace invariant. Thus, the  $\Gamma_j$  measure the inverse relaxation or equilibration times. The expression for  $\Sigma^2(\tau)$  incorporating this result is given in the appendix; again the integrals are cutoff when  $b(\tau)$  first attains 0.

The essential content of equation (2.17) can be seen with the result for  $N = 2$ .

$$I(\tau) = 1 + e^{-\Gamma\tau} \quad (2.18)$$

where  $\Gamma = \gamma_{12}(V_1^{-1} + V_2^{-1})$ . Substituting this result into equation (2.10) gives

$$1 - b(\tau) = 2\tau(1 + e^{-\Gamma\tau}). \quad (2.19)$$

The behaviour of  $b(\tau)$  now depends on the interplay of  $\tau$  and  $\Gamma$ . We have

$$1 - b(\tau) \approx \begin{cases} 2 \cdot 2\tau & \tau\Gamma \ll 1 & \tau < \frac{1}{4} \\ 2\tau & \tau\Gamma \gg 1. \end{cases} \quad (2.20)$$

In words, for times shorter than the equilibration time, the system's form factor behaves as though the spectrum is composed of two independent GOE spectra superposed randomly. At the other extreme ( $\tau$  much longer than  $\Gamma^{-1}$ ), the system recovers the form factor of a single GOE. Equation (2.19) governs the transitional behaviour as well. This behaviour of the form factor is valid only in the semiclassical regime  $\tau \ll 1$ . For  $\tau$  beyond this, the statistical arguments start to breakdown and  $b(\tau)$ 's approach to zero is not properly treated; see the discussion of the crossover time in [4] and note that superposing independent spectra reduces the crossover time by renormalizing  $\tau$  with the density of states. Due to this difficulty, the small level spacing fluctuations are not expected to be accurately reflected in the theory presented here (even if the semiclassical trace formula itself was capable of handling small spacings).

As a final comment, note that a bounded system composed of a larger and larger number of subsystems must resemble more and more a system with some diffusion length scale with the diffusion coefficient being a function of the connecting fluxes. Therefore, the theory presented here must recover what is known about the

fluctuations of diffusive systems as  $N \rightarrow \infty$ . In the simple case of a linear chain of  $N$  ( $\gg 1$ ) subsystems connected only to their nearest neighbours, the diffusive character of the system can be easily shown. For simplicity assume the subsystems are aperiodic, but uniform or translationally invariant statistically speaking. This avoids the necessity of coherent summing of periodic orbit contributions; see [10] for the crossover time in the more general problem. If the distributions of the domain volumes and the fluxes are uniform,  $\text{Tr}(e^{-M\tau})$  can be found analytically by a number of methods. Letting  $\gamma$  denote the transition probability in scaled time units and using periodic boundary conditions gives

$$\begin{aligned} \text{Tr}(e^{-M\tau}) &\approx Ne^{-2\gamma\tau} I_0(2\gamma\tau) \\ &\sim N\sqrt{1/4\pi\gamma\tau} \end{aligned} \quad (2.21)$$

where  $I_0(x)$  ( $= J_0(ix)$ ) is a modified Bessel function of the first kind. This is valid for times such that the diffusion constant multiplied by the time give a length scale shorter than the system's length. Equation (2.21) inserted into equation (2.10) leads to a  $\tau^{1/2}$  dependence for the form factor as discussed by Dittrich and Smilansky [10]. (The extension to diffusion in more dimensions, and some numerical tests can be found in [18].)

### 3. Application to the quartic oscillator

A good testing ground for this theory is provided by the system of two coupled, quartic oscillators studied by BTU [12]. From their work, we have already available both a long stretch of a quantum spectrum as well as all the classical information required to apply the results of section 2. The Hamiltonian they investigated is

$$H(\lambda, b) = \frac{p^2}{2} + a(\lambda) \left( \frac{q_1^4}{b} + bq_2^4 + 2\lambda q_1^2 q_2^2 \right). \quad (3.1)$$

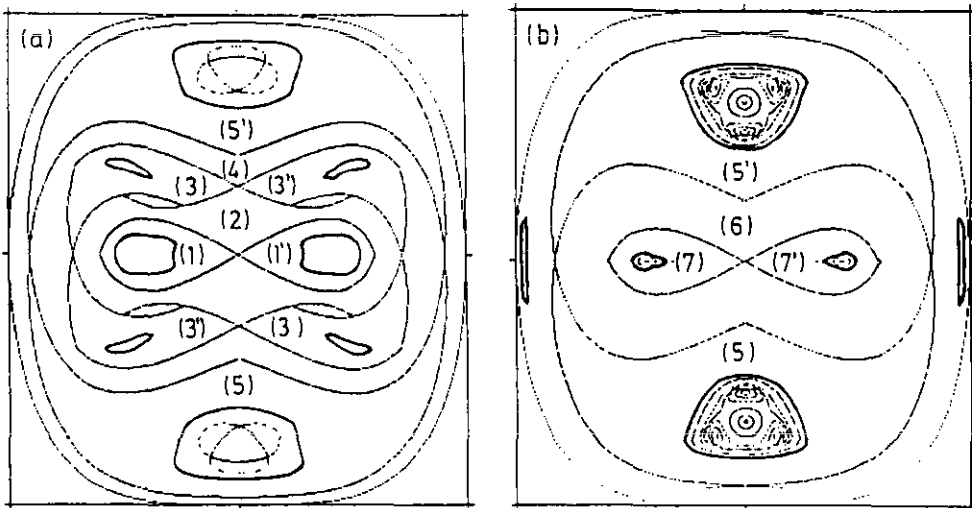
One advantage of this system is that the homogeneity of the potential greatly simplifies the classical and semiclassical aspects of the problem since the effect of changing the energy is just a rescaling of the orbits. For example, the classical actions scale as  $E^{3/4}$ . The nature of the dynamics is determined by the magnitude of the coupling between the modes,  $\lambda$ . When it is zero the system is integrable and as it is decreased toward  $-1$ , the system becomes progressively more chaotic.  $b$  was introduced to lower the symmetry from that of a square to that of a rectangle, and the adjustable constant  $a(\lambda)$  was inserted for reasons of simplifying the quantum calculations.

BTU calculated the quantum and classical mechanics of the case  $(\lambda, b) = (-0.35, \pi/4)$  in detail. They obtained to high precision the first 22 000 eigenvalues giving the possibility of calculating very accurate fluctuation measures; the large majority of the energy levels were obtained to better than  $10^{-5}$  of a mean spacing. The classical dynamics is a little more complicated. The system contains both regular and chaotic regions of phase space. This particular example is approximately 88% chaotic by relative phase space volume and approximately 12% regular. The regular dynamics, not being of interest here, requires a means for removing its effects. BTU made use of the discrete symmetries of the Hamiltonian to classify all the states into a regular or chaotic group; all the tori occur in pairs and a near degeneracy is implied



for all the regular states and no others. The eigenvalues corresponding to the regular region (all the quantizing tori) could then be lifted out of the spectrum leaving only those levels associated with the chaotic region; the percentage of eigenvalues removed matched well the 12% expectation.

There are no configuration space barriers in the quartic oscillator as in the example of a string of stadia connected by windows. Instead, there are phase-space mechanisms inhibiting transport in the form of broken separatrices. Each has a turnstile which allows only a limited flux of phase points to cross from one side to the other. An attractive feature of these barriers is the absence of corners or cusps which one often encounters in problems which involve billiards (wave guides). Of all the structure in the chaotic domain, only four barriers introduce time scales significant enough to require consideration for the level statistics. Each of the five subsystems thus implied are determined by defining on which side of the separatrices they are found (for details see [12]). Seven subregions are delineated in the surfaces of section plotted in figure 1. However, due to the rapid communication implied by the fluxes, the regions numbered 1 and 2 effectively constitute only a single region, and similarly for 6 and 7. With this taken into account, the respective volumes and fluxes are given in table 1.



**Figure 1.** The various, weakly communicating subregions in phase space separated by partial transport barriers: (a)  $q_1 = 0$  Poincaré section and (b)  $q_2 = 0$  Poincaré section. Both surfaces are needed to show all of the regions, even though the outer perimeter of one section roughly corresponds to the inner region of the other section. The KAM islands (regular motion) and some of their structure is also shown. Due to the discrete symmetries of the problem, a region and its symmetric primed region constitute only one region from the point of view of the quantum system within a specified symmetry representation. Taken from the second letter of reference [12].

The spectral statistics will be calculated separately for each symmetry representation before being combined to give an average behaviour. Therefore, we need to desymmetrize the classical system which can be done by placing a hard wall in both the  $q_1 = 0$  and  $q_2 = 0$  planes. The effect is to reduce the fluxes given in table 1 by a factor four. As viewed from the surface of section, one factor two comes from the disappearance of each turnstile's symmetric partner. The other arises because

**Table 1.** Relative volume and connecting fluxes of the chaotic phase space subregions as shown in figure 1. The fluxes are calculated for  $E = 1$  and must be scaled as  $E^{3/4}$ . Regions 1+2 and 6+7 were grouped together because their connecting fluxes are such that their barriers could be ignored. Taken from the second letter of [12].

Region	Relative volume (%)	Area	Total flux
1+2	12	1.1	(1+2) $\leftrightarrow$ 3 0.068
3	13	1.2	3 $\leftrightarrow$ 4 0.13
4	13	1.2	4 $\leftrightarrow$ 5 0.21
5	36	2.6	5 $\leftrightarrow$ (6+7) 0.28
6+7	26	2.1	

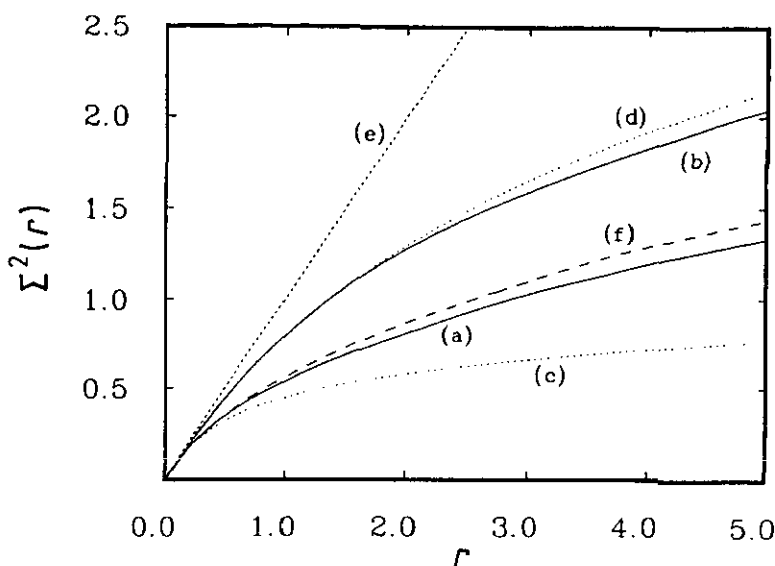
before desymmetrization a phase point hopped across two gates in the turnstile at each iteration of the map, but afterward only one (the phase points can be thought of as striking the desymmetrized surface of section twice as often). The reduction of four in the total phase-space volume is automatically accounted for in the scale of  $\tau$  and the relative volumes are left unchanged. Thus overall, the  $\Gamma_j$  are reduced by a factor four.

With table 1, the  $M$  matrix is easily constructed and diagonalized. Due to the homogeneity of the potential we may obtain the  $\Gamma_j$  at any energy and scale them appropriately to find the results for any other energy. We can also predict the energy dependence of the fluctuations easily. In this example, a stretch of  $\sim 6000$  levels (the 16000th to 22000th levels) is used to calculate  $\Sigma^2(r)$ . The  $\Gamma_j$  are  $\Gamma_1 = 19.67$ ,  $\Gamma_2 = 11.00$ ,  $\Gamma_3 = 6.56$ ,  $\Gamma_4 = 1.70$ , and  $\Gamma_5 = 0.0$  for a mean energy equal to 706, thus fixing the prediction for the statistics (using the equation in the appendix). In figure 2, we show several  $\Sigma^2(r)$  curves for comparison. In addition to the quartic oscillator results, the BTU prediction and our semiclassical prediction, are the random matrix ensemble result expected for a single chaotic zone (as though the transport barriers were easily crossed) and also for five disconnected zones (as though the barriers could not be crossed at all). Of course, the quartic oscillator lies intermediate between these extremes. A final curve, that being Poissonian, can be thought of as the limit of a large number of non-communicating regions.

Before passing judgment on the semiclassical analysis, it is necessary to comment on our expectations. The leading large- $r$  behaviour of  $\Sigma^2(r)$  for a single GOE is

$$\Sigma^2(r) \sim a \ln r + b \quad (3.2)$$

where  $a = 2/\pi^2$ . For  $N$  randomly superposed GOE, the same formula applies with  $a$  replaced by  $Na$ . The semiclassical derivation only yields  $a$  (or  $Na$ ) exactly and not  $b$ . The root cause is the interpolation difficulty mentioned at the end of section 2.2. In fact, if we evaluate the error in  $b$  analytically for the case of five regions weighted as given by the relative volumes in table 1 but taking the fluxes equal to zero, the semiclassical determination of  $b$  is too large by 0.654. For a single GOE, the error is 0.110. So all that can be anticipated from the semiclassical treatment of  $\Sigma^2(r)$  is that the large- $r$  behaviour parallels the quartic oscillator's and not that there is absolute agreement. It also seems plausible that the large- $r$  difference between the



**Figure 2.** Number variance,  $\Sigma^2(\tau)$ : (a) the quantum spectral fluctuations from the 16 000th to the 22 000th levels; (b) semiclassical prediction of section 2.3; (c) single GOE; (d) five randomly superposed GOE spectra weighted according to the appropriate fractional phase-space volumes given in table 1; (e) Poissonian result; and (f) generalized ensemble prediction given by BTU in [12]. See text for further explanation.

two should approach a constant somewhere in a range between the errors in  $b$  at the two extrema mentioned just above. This is seen to be the case in figure 2.

#### 4. Discussion and conclusions

The simplest classically mixing systems with a finite phase-space volume achieve statistical equilibrium after a short time independently of the state in which they are initially prepared. The equilibrium distribution is the phase-space invariant measure. We can characterize these distributions by a single parameter—the phase-space volume and the observation that the distribution is invariant under canonical transformations. This way of describing the equilibrium state of *classically* chaotic systems is the direct analogue of describing *quantum* systems in terms of random matrix theory, where the ensemble is completely determined once the mean level density is prescribed, together with invariance of the measure under appropriate coordinate transformations (the dynamics of the ensembles imply essentially instantaneous complete mixing). The analogy becomes complete through the proportionality of the phase-space volume and the mean level density (equation (2.6)). This analogy is borne out in the semiclassical derivation of the spectral form factor  $b(\tau)$  (equation (2.12)), where the only requirement on the classical dynamics was that an equipartition is reached after a sufficiently short time. Not all systems can be characterized in terms of a fast mixing stage followed by equilibrium distributions. We have considered a larger class of systems with isolated transport barriers which introduce new time scales. The classical and quantum mechanical descriptions are affected in a complementary way as predicted by the semiclassics and demonstrated with the coupled quartic oscillators. In the limit

that the number of barriers is fixed, each becoming more and more open, the random matrix results for the rapid, fully mixing case are recovered. Though not surprising, understanding the transition to the behaviour of one of the canonical ensembles in this way better defines the conditions to be met before invoking these ensembles.

Composite systems also include as a limiting subclass, systems with the number of barriers large and tending to infinity. If only the neighbouring regions directly communicate, an infinite chain (lattice in more dimensions) is formed. The classical mixing motion within each constituent along with the transport leads to a global diffusive evolution, and due to the infinite volume of the phase space, equilibration is never achieved. A new time scale—the diffusion time now describes the equilibration process, and it induces essentially new features in the corresponding spectral fluctuations. The proportionality of  $b(\tau)$  to  $\tau^{1/2}$  instead of  $\tau$  is one direct consequence.

This work complements the image of 'semiclassical' generalized ensembles constructed in the theory of BTU. These ensembles have not yet been treated analytically in order to obtain their predictions and Monte Carlo calculations are still needed. It would be interesting to know to what extent the semiclassical theory given here would be consistent with the leading order asymptotic solution to these ensembles. Assuming this consistency and that the BTU ensembles would lead to banded random matrices for diffusive systems, one would begin to have a dynamical image of the relevance and asymptotic nature of banded ensembles for diffusive systems.

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### Appendix

With equations (2.5), (2.10) and (2.17), the number variance  $\Sigma^2(r)$  is found to be

$$\begin{aligned} \Sigma^2(r) = r \left( 1 - \frac{2}{\pi} \text{Si}(2\pi r \tau_m) \right) + \frac{1 - \cos(2\pi r \tau_m)}{\pi^2 \tau_m} \\ + \frac{2}{\pi^2} (\gamma + \ln(2\pi r \tau_m) - \text{Ci}(2\pi r \tau_m)) + \frac{1}{\pi^2} \sum_{j=1}^{N-1} \left( \ln \left( 1 + \left( \frac{2\pi r}{\Gamma_j} \right)^2 \right) \right. \\ \left. - 2E_1(\Gamma_j \tau_m) + 2\text{Real} \{ E_1(\Gamma_j \tau_m + 2\pi i r \tau_m) \} \right) \end{aligned} \quad (\text{A.1})$$

where  $E_1(z)$ ,  $\text{Si}(z)$  and  $\text{Ci}(z)$  are the exponential integral, sine integral and cosine integral functions, respectively. The  $\Gamma_j$  are those rescaled to the natural units of the problem (multiplied by  $2\pi\hbar\langle\rho\rangle_E$ ). The quantity  $\tau_m$  is taken as the point where  $b(\tau)$  first attains the value 0. It was located by setting up a recursive method.

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