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A perturbative multiple scattering theory

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Abstract

We extend existing multiple scattering theory to consider cases in which some scatterers are removed, shifted, or replaced. The study helps us understand the stability of scattering systems, and provides a powerful analytical tool for treating perturbation in few-body scattering experiments, particularly in atomic and mesoscopic physics.

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1. Introduction

Scattering theory has been well studied in the past century [1]. Initially developed to help study atomic and nuclear structures, scattering theory was soon also applied to condensed matter physics, and then went beyond quantum scattering on microscopic and mesoscopic scales to find applications in classical, macroscopic systems in optics [2,3], and more recently in acoustic scattering experiments [4,5]. As Maynard [6] pointed out, though quantum scattering is derived from the Schrödinger equation and classical scattering is based on a classical wave equation, many analogies between them exist under appropriate conditions. However, it remains to be seen what impact a small perturbation has on a scattering system. A perturbation theory can give important insight into the system's internal dynamics, which are often too difficult to analyze directly.

Our work is motivated by the recent experimental and theoretical studies on perturbations in acoustic time reversal focusing [7,8]. As the first part of our ongoing project on perturbed time reversal focusing, this Letter gives an analytical formulation of perturbations in multiple scattering, applicable to both quantum and classical systems [6]. A following piece of work [9] will explore perturbations in acoustic time reversal focusing in great detail, and not only provide an analytical model to reproduce the observations [7], but demonstrate analogous features in quantum scattering systems as well. By developing a perturbative multiple scattering theory in a general form in this Letter, we expect its future broad applications to both theoretical study of stability of scattering systems and experimental research in atomic and mesoscopic physics. For example, studies on photon interaction with

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atoms, atomic collision in Bose–Einstein condensates, and electronic transport in mesoscopic/nanoscopic systems can all benefit from an understanding of the influences of small changes on scatterers (atoms or impurities) due to perturbations like temperature fluctuations, etc. Also the knowledge of perturbation effects could be used to design apparatus to monitor gradual changes over long time in various systems, such as the proposed industrial/medical imaging and wireless communication schemes based on time reversal focusing [7].

The Letter is organized as follows. First we review the conventional formulation of multiple scattering systems. Then the change in the overall incoming wave on an arbitrary scatterer is derived in terms of the unperturbed wave field for removing a scatterer, shifting a scatterer, and replacing a scatterer with a different one, respectively. In the end, the collective perturbation due to changes on multiple scatterers is given in a compact form. Physical interpretation is given along with mathematical derivations. All formulations are given for a general multiple scattering system. For simplicity, only s-wave scattering is considered, since it's the most commonly explored case in multiple scattering research and can be easily obtained in mesoscopic systems by lowering temperatures to achieve wavelengths comparable to scatterer size.

2. The conventional s-wave multiple scattering theory

Consider an open, linear, scalar-wave system consisting of a source and a number of s-wave scatterers whose scattering strength is uniform and given by $\epsilon(\omega)$. Denote the source's and N scatterers' positions, respectively, as $\vec{x}_s$ and $\{\vec{x}_i; i = 1, N\}$. From an s-wave multiple scattering theory [1], the total resulting wave field $\phi^N(\vec{x};\omega)$ at an arbitrary point $\vec{x}$ can be expressed in position-frequency space using the free propagation Green function $G(\vec{x},\vec{x}';\omega)$ as

$$\phi^N(\vec{x};\omega) = \phi_s(\vec{x};\omega) + \epsilon(\omega)\sum_{i=1}^{N} G(\vec{x},\vec{x}_i;\omega)\phi^N(\vec{x}_i;\omega) = \phi_s(\vec{x};\omega) + \epsilon(\omega)\sum_{i=1}^{N} G_{\vec{x},i}\phi_i^N, \tag{1}$$

where $\phi_i^N = \phi^N(\vec{x}_i;\omega)$, $G_{\vec{x},i} = G(\vec{x},\vec{x}_i;\omega)$, $\phi_s(\vec{x};\omega)$ is the unscattered signal directly from the source, $\phi^N(\vec{x}_i;\omega)$ is the overall incoming wave at the ith scatterer, and the superscript N denotes the total number of scatterers present.

Setting $\vec{x} = \vec{x}_i$ successively in Eq. (1) for all i leads to a series of equations that can be expressed formally in the matrix representation

$$M^N \cdot \phi^N \equiv \begin{pmatrix} m_{11} & \cdots & m_{1N} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ m_{N1} & \cdots & m_{NN} \end{pmatrix} \begin{pmatrix} \phi_1^N \\ \cdot \\ \cdot \\ \phi_N^N \end{pmatrix} = \begin{pmatrix} \phi_{1,s} \\ \cdot \\ \cdot \\ \phi_{N,s} \end{pmatrix}, \tag{2}$$

where $\phi_{i,s} = \phi_s(\vec{x}_i;\omega)$, $m_{ii} = 1$, and $m_{ij} = -\epsilon(\omega)G_{i,j}$ for $i \neq j$. The standard method of solving heterogeneous matrix equations gives

$$|M^N|\phi_i^N = \begin{vmatrix} m_{11} & \cdot & m_{1(i-1)} & \phi_{1,s} & m_{1(i+1)} & \cdot & m_{1N} \\ m_{21} & \cdot & m_{2(i-1)} & \phi_{2,s} & m_{2(i+1)} & \cdot & m_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ m_{N1} & \cdot & m_{N(i-1)} & \phi_{N,s} & m_{N(i+1)} & \cdot & m_{NN} \end{vmatrix}. \tag{3}$$

The determinant on the right-hand side is obtained by replacing the ith column with the vector $\{\phi_{i,s}\}$. From the standard matrix theory [10], the determinant $|M^N|$ can be written as $\sum_p \pm[m_{p(1)1}m_{p(2)2}m_{p(3)3}\cdots m_{p(N)N}]$, where the sum is extended over all permutations p of the integers $1, 2, \ldots, N$ and a $+$ or $-$ sign is affixed to each product according to whether p is even or odd for $p(1), p(2), \ldots, p(N)$. Given that, for $i \neq j$, m_{ij} describes free wave propagation from the jth scatterer toward the ith scatterer, any term $m_{p(1)1}m_{p(2)2}m_{p(3)3}\cdots m_{p(N)N}$ can be reordered to have indices linked as $m_{jl}m_{lh}m_{hk}m_{ks}m_{st}\cdots m_{pj}$ after removing factors like m_{hh} ($= 1$, which means

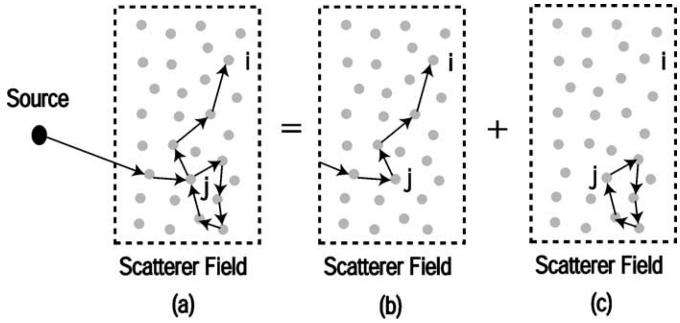

Fig. 1. (a) Diagram of a typical scattering path from the source to the ith scatterer, which visits the jth scatterer twice. The entire path can be decomposed into two parts described in (b) and (c), respectively. (b) The basic component of the scattering path which visits every scatterer no more than once. (c) The loop component of the scattering path that begins and ends at the jth scatterer. There can be none or multiple loops in an arbitrary scattering path.

the hth scatterer isn't involved in scattering). Each such term corresponds to a unique scattering path that visits every scatterer at most once and that starts and ends at the jth scatterer, i.e., a closed loop as the one in Fig. 1(c). So $|M^N|$ describes all kinds of possible loops in scattering paths. Similarly, the right-hand side determinant of Eq. (3) is the sum of terms each of which can be reordered into the form $m_{ij}m_{jh}m_{hk}m_{ks}m_{st}\cdots m_{pq}\phi_{q,s}$, i.e., the index begins at i and finishes at s on the right-hand side. Each term corresponds to a unique scattering path that visits every scatterer at most once, starting from the source and ending at the ith scatterer as the one in Fig. 1(b). Now a scattering path interpretation to Eq. (3) is manifest. Since ϕ_i^N describes the overall incoming wave on the ith scatterer from source, it contains contributions from the various scattering paths from source to the scatterer. Any of the various paths, such as the one in Fig. 1(a), can be decomposed into one basic path that visits every scatterer at most once, as the one in Fig. 1(b), and a possible second part that consists of one or more loops each of which starts and ends at an intermediate scatterer on the basic path, as the one in Fig. 1(c). Interestingly, the right- and left-hand sides of Eq. (3) describe all possible choices of basic paths and loops that may be involved in scattering, thereby offering a convenient and intuitive way to explain later results.

One should notice that if $|M^N|$ is small, ϕ_i^N will be large for the ith scatterer with a non-trivial right-hand side determinant in Eq. (3), which usually corresponds to certain resonance of the system. The form in Eq. (2) is particularly useful for generating approximations to the changes in the wave field due to removing, shifting, or changing the strengths of some of the scatterers. For most applications, the natural next step is to invert the $\{m_{ij}\}$ matrix in Eq. (2) or calculate the left- and right-hand side determinants in Eq. (3) to obtain ϕ_i^N, almost always done numerically. But here we stay with the matrix form to develop analytical formulas for the various perturbations in the following sections.

3. Removing an s-wave scatterer

As a starting point, let us focus on the changes that will be introduced by removing an arbitrary scatterer labeled by the index N. According to Eq. (1), the change in the wave field is

$$\delta\phi^N(\vec{x};\omega) \equiv \phi^{N-1}(\vec{x};\omega) - \phi^N(\vec{x};\omega) = \epsilon(\omega)\left(\sum_{i=1}^{N-1} G_{\vec{x},i}\delta\phi_i^N - G_{\vec{x},N}\phi_N^N\right), \tag{4}$$

where $\delta\phi_i^N = \phi_i^{N-1} - \phi_i^N$ for $1 \leqslant i \leqslant N-1$ is the change in the overall incoming wave on the remaining ith scatterer. For later convenience, here we define $\delta\phi_N^N = \phi_{N,s} - \phi_N^N$. Eq. (4) expresses the change in the wave field as the sum of the propagated changes from the remaining scatterers minus the propagated field from the absent scatterer. An expression for the $\{\delta\phi_i^N\}$ is sought in terms of the unperturbed or N-scatterer wave field quantities.

For the wave field generated with just the first $N-1$ scatterers, a similar matrix representation exists as the N-scatterer case except with one fewer row and column. Thus,

$$M^{N-1}\cdot\phi^{N-1}\equiv\begin{pmatrix} m_{11} & \cdots & m_{1(N-1)} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ m_{(N-1)1} & \cdots & m_{(N-1)(N-1)} \end{pmatrix}\begin{pmatrix}\phi_1^{N-1}\\ \cdot \\ \cdot \\ \phi_{N-1}^{N-1}\end{pmatrix}=\begin{pmatrix}\phi_{1,s}\\ \cdot \\ \cdot \\ \phi_{N-1,s}\end{pmatrix}. \tag{5}$$

Substituting the left-hand side of Eq. (5) into the first $N-1$ elements of the vector $\{\phi_{i,s}\}$ on the right-hand side of Eq. (2) and subtracting $\begin{pmatrix} M^{N-1} & \mathbf{0} \\ \mathbf{0} & 1\end{pmatrix}\cdot\phi^N$ from both sides leads to a solvable series of equations

$$\begin{pmatrix} M^{N-1} & \mathbf{0} \\ \mathbf{0} & 1\end{pmatrix}\begin{pmatrix}\delta\phi_1^N\\ \cdot \\ \cdot \\ \delta\phi_N^N\end{pmatrix}=\begin{pmatrix} 0 & \cdots & 0 & m_{1N} \\ \cdot & \cdots & \cdot & \cdot \\ 0 & \cdots & 0 & m_{(N-1)N} \\ m_{N1} & \cdots & m_{N(N-1)} & 0\end{pmatrix}\begin{pmatrix}\phi_1^N\\ \cdot \\ \cdot \\ \phi_N^N\end{pmatrix}=\begin{pmatrix} m_{1N} \\ \cdot \\ \cdot \\ m_{(N-1)N} \\ m'_{NN}\end{pmatrix}\phi_N^N, \tag{6}$$

where $m'_{NN}=\frac{1}{\phi_N^N}\sum_{i=1}^{N-1}m_{Ni}\phi_i^N$. The changes in the wave field due to removing a single scatterer now appear only on the left-hand side of the equation, whereas the right-hand side contains only the original wave field quantities. Using the same method of solution of matrix equations as before gives for $1\leqslant i\leqslant N-1$

$$\left|M^{N-1}\right|\delta\phi_i^N=\begin{vmatrix} m_{11} & \cdot & m_{1(i-1)} & m_{1N} & m_{1(i+1)} & \cdot & m_{1(N-1)} \\ m_{21} & \cdot & m_{2(i-1)} & m_{2N} & m_{2(i+1)} & \cdot & m_{2(N-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ m_{(N-1)1} & \cdot & m_{(N-1)(i-1)} & m_{(N-1)N} & m_{(N-1)(i+1)} & \cdot & m_{(N-1)(N-1)}\end{vmatrix}\phi_N^N, \tag{7}$$

where both determinants are of $(N-1)\times(N-1)$ matrices. The right-hand side is obtained by first replacing the ith column with the Nth column, and next removing the Nth row and column from M^N. Now the right-hand side terms are like $m_{ij}m_{jh}m_{hk}m_{ks}m_{st}\cdots m_{pN}\phi_N^N$, i.e., the index begins at i and finishes at N. Given the ordering sequence of the terms in the expanded determinants has a path interpretation, one sees that, after the Nth scatterer is removed, the information of ϕ_N^N's absence can reach the ith scatterer through paths described on the right-hand side, with possible loops at intermediate scatterers given by the left-hand side terms as shown in Fig. 2. To obtain $\delta\phi_i^N$, one can numerically solve either Eq. (6) or Eq. (7). The sum of the various terms in either of the two determinants of Eq. (7) strongly depends on the convergence of high order scattering terms.

Before proceeding to a different perturbation, let us examine the implications of Eq. (7). The left-hand side $|M^{N-1}|$ is usually non-zero for real frequencies in non-absorbing medium. The i-dependent right-hand side determinant must be non-zero for at least part of the remaining scatterers. Therefore when $|M^{N-1}|$ is very small, $\delta\phi_i^N$ will be large for the ith remaining scatterer with a non-trivial right-hand side determinant. In such situation, removing the Nth scatterer could either induce a resonance in a scattering system or suppress an existing resonance if $|M^N|$ is small in Eq. (3).

Since each m_{ij} $(i\neq j)$ contains one $\epsilon(\omega)$, in the weak scattering limit in a ballistic transport system (such as a clean quantum wire), it's possible to obtain an approximate analytical solution to $\delta\phi_i^N$ using only the few lowest order terms in Eq. (7). For example, keeping up to the first order terms in $\epsilon(\omega)$ in Eq. (7) leads to $\delta\phi_i^N\approx m_{iN}\phi_N^N=-\epsilon(\omega)G(\vec{x}_i,\vec{x}_N;\omega)\phi_N^N$. Thus, among the remaining scatterers, the amplitude of $\delta\phi_i^N$ varies in proportion to that of $G(\vec{x}_i,\vec{x}_N;\omega)$, which is, for $k|\vec{x}_i-\vec{x}_N|\gg 1$, $2\pi/k$ in 1D; $\sqrt{\frac{2\pi}{k|\vec{x}_i-\vec{x}_N|}}$ in 2D; and $1/|\vec{x}_i-\vec{x}_N|$ in 3D ($k=2\pi/\lambda$ and λ is the wavelength) [11]. This implies that in the weak scattering limit the amplitude of $\delta\phi_i^N$ is not only small, but non-increasing in distance $|\vec{x}_i-\vec{x}_N|$ from the perturbed scatterer. But in the strong scattering

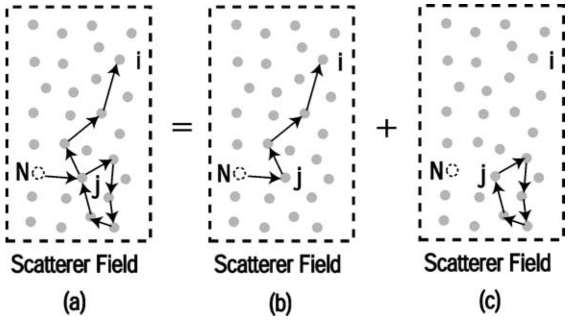

Fig. 2. (a) Diagram of a typical path for the absence of the Nth scatterer to reach the ith scatterer, which visits the jth scatterer twice. The entire path can be decomposed into two parts described in (b) and (c), respectively. (b) The basic component of the path which visits every scatterer no more than once. (c) The loop component of the path that begins and ends at the jth scatterer. There can be none or multiple loops in an arbitrary path.

limit, such a simple relationship no longer holds and the amplitude of $\delta\phi_i^N$ could be larger in far field than near field given appropriate conditions.

4. Shifting a scatterer

Analogous equations can be developed to account for the effect of displacing a scatterer. The whole process can be viewed as a two-step operation. First, remove the Nth scatterer from its original position, and second, place it in its new location. The first step can be described by the same set of equations developed above for removing a scatterer. The reverse process of the second step is to remove the scatterer from its new location, which is also described by the same set of equations except that ϕ_N^N, $\delta\phi_i^N$, and m_{iN} in Eq. (7) are replaced by primed versions for the new location of the Nth scatterer. The total effect follows by subtraction of the two sets of equations. Letting $\delta\phi_i^N$ $(i < N)$ denote the full change of the compound process $\delta\phi_i^N = (\phi_i^N)' - \phi_i^N$ gives

$$\left|M^{N-1}\right|\delta\phi_i^N = \begin{vmatrix} m_{11} & \cdot & m_{1(i-1)} & m_{1N} & m_{1(i+1)} & \cdot & m_{1(N-1)} \\ m_{21} & \cdot & m_{2(i-1)} & m_{2N} & m_{2(i+1)} & \cdot & m_{2(N-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ m_{(N-1)1} & \cdot & m_{(N-1)(i-1)} & m_{(N-1)N} & m_{(N-1)(i+1)} & \cdot & m_{(N-1)(N-1)} \end{vmatrix} \phi_N^N - \begin{vmatrix} m_{11} & \cdot & m_{1(i-1)} & m_{1N'} & m_{1(i+1)} & \cdot & m_{1(N-1)} \\ m_{21} & \cdot & m_{2(i-1)} & m_{2N'} & m_{2(i+1)} & \cdot & m_{2(N-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ m_{(N-1)1} & \cdot & m_{(N-1)(i-1)} & m_{(N-1)N'} & m_{(N-1)(i+1)} & \cdot & m_{(N-1)(N-1)} \end{vmatrix} \phi_{N'}^N, \tag{8}$$

where $\phi_{N'}^N$ is the overall incoming wave at the Nth scatterer when placed at its new location. Note that the two determinants on the right-hand side are not identical because of their dependence on the Nth scatterer's locations.

But Eq. (8) is not the best way to express the changes since it involves $\phi_{N'}^N$, which isn't present in the original wave field. A better solution is to first write an equation similar to Eq. (2) for the new wave field

$$\left(M^N\right)' \cdot \left(\phi^N\right)' \equiv \begin{pmatrix} m_{11} & \cdots & m_{1(N-1)} & m_{1N'} \\ \cdot & \cdots & \cdot & \cdot \\ m_{(N-1)1} & \cdots & m_{(N-1)(N-1)} & m_{(N-1)N'} \\ m_{N'1} & \cdots & m_{N'(N-1)} & m_{N'N'} \end{pmatrix} \begin{pmatrix} (\phi_1^N)' \\ \cdot \\ \cdot \\ (\phi_N^N)' \end{pmatrix} = \begin{pmatrix} \phi_{1,s} \\ \cdot \\ \cdot \\ \phi_{N',s} \end{pmatrix}, \tag{9}$$

which, when compared with Eq. (2), gives

$$\left(M^N\right)' \cdot \left(\phi^N\right)' = \begin{pmatrix} 0 \\ \cdot \\ 0 \\ \phi_{N',s} - \phi_{N,s} \end{pmatrix} + M^N \cdot \phi^N. \tag{10}$$

Defining $\delta\phi_i^N = (\phi_i^N)' - \phi_i^N$ for all i and subtracting $(M^N)' \cdot \phi^N$ from both sides leads to

$$\begin{pmatrix} m_{11} & \cdots & m_{1(N-1)} & m_{1N'} \\ \cdot & \cdots & \cdot & \cdot \\ m_{(N-1)1} & \cdots & m_{(N-1)(N-1)} & m_{(N-1)N'} \\ m_{N'1} & \cdots & m_{N'(N-1)} & m_{N'N'} \end{pmatrix} \begin{pmatrix} \delta\phi_1^N \\ \cdot \\ \cdot \\ \delta\phi_N^N \end{pmatrix} = \begin{pmatrix} m_{1N} - m_{1N'} \\ \cdot \\ m_{(N-1)N} - m_{(N-1)N'} \\ \Delta m_{NN} \end{pmatrix} \phi_N^N, \tag{11}$$

where $\Delta m_{NN} = [(\phi_{N',s} - \phi_{N,s}) + \sum_{i=1}^{N-1} (m_{Ni} - m_{N'i})\phi_i^N]/\phi_N^N$. The equations can be solved to give for all i

$$\begin{vmatrix} m_{11} & \cdots & m_{1(N-1)} & m_{1N'} \\ \cdot & \cdots & \cdot & \cdot \\ m_{(N-1)1} & \cdots & m_{(N-1)(N-1)} & m_{(N-1)N'} \\ m_{N'1} & \cdots & m_{N'(N-1)} & m_{N'N'} \end{vmatrix} \delta\phi_i^N$$
$$= \begin{vmatrix} m_{11} & \cdot & m_{1(i-1)} & m_{1N} - m_{1N'} & m_{1(i+1)} & \cdot & m_{1N'} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ m_{(N-1)1} & \cdot & m_{(N-1)(i-1)} & m_{(N-1)N} - m_{(N-1)N'} & m_{(N-1)(i+1)} & \cdot & m_{(N-1)N'} \\ m_{N'1} & \cdot & m_{N'(i-1)} & \Delta m_{NN} & m_{N'(i+1)} & \cdot & m_{N'N'} \end{vmatrix} \phi_N^N. \tag{12}$$

Now only the original wave field quantities and $m_{iN'}/m_{N'i}$ (related to the new location of the Nth scatterer and thus necessarily included) are used to express the changes in the overall incoming wave on the ith scatterer. Moreover, Eq. (12) gives $\delta\phi_i^N$ for all i, including the shifted Nth scatterer itself.

Similar to the preceding section, if $|(M^N)'|$ is small, $\delta\phi_i^N$ will be large for the ith scatterer with a non-trivial right-hand side determinant in Eq. (12). In such situation, shifting the Nth scatterer could either induce a resonance in a scattering system, or suppress an existing resonance if $|M^N|$ is small in Eq. (3).

5. Changing a scatterer's strength

It is also straightforward to apply a similar method to the case in which the Nth scatterer changes its scattering strength by a factor of an arbitrary constant γ. This can be viewed as a special case of Eq. (8) by replacing $m_{iN'}$ with γm_{iN} in the second determinant on the right-hand side. One should note that, for $i \neq N$, it's m_{iN} instead of m_{Ni} that is multiplied by γ (m_{Ni} describes a scattering by the ith scatterer toward the Nth scatterer), and that $\phi_{N'}^N \neq \phi_N^N$ (despite the same position for the scatterer) because the change in scattering strength of one scatterer modifies the wave field at every point. However, as in the preceding section, there is a better expression for the changes in wave field not involving $\phi_{N'}^N$. Similar to Eq. (2), the new wave field has

$$\left(M^N\right)'' \cdot \left(\phi^N\right)'' \equiv \begin{pmatrix} m_{11} & \cdots & m_{1(N-1)} & \gamma m_{1N} \\ \cdot & \cdots & \cdot & \cdot \\ m_{(N-1)1} & \cdots & m_{(N-1)(N-1)} & \gamma m_{(N-1)N} \\ m_{N1} & \cdots & m_{N(N-1)} & m_{NN} \end{pmatrix} \begin{pmatrix} (\phi_1^N)'' \\ \cdot \\ \cdot \\ (\phi_N^N)'' \end{pmatrix} = \begin{pmatrix} \phi_{1,s} \\ \cdot \\ \cdot \\ \phi_{N,s} \end{pmatrix}. \tag{13}$$

Compared with Eq. (2), Eq. (13) gives

$$\begin{aligned}\left(M^N\right)''\cdot\left(\phi^N\right)'' &\equiv \begin{pmatrix} m_{11} & \cdots & m_{1(N-1)} & \gamma m_{1N} \\ \cdot & \cdots & \cdot & \cdot \\ m_{(N-1)1} & \cdots & m_{(N-1)(N-1)} & \gamma m_{(N-1)N} \\ m_{N1} & \cdots & m_{N(N-1)} & m_{NN} \end{pmatrix}\begin{pmatrix} (\phi_1^N)'' \\ \cdot \\ \cdot \\ (\phi_N^N)'' \end{pmatrix} \\ &= \begin{pmatrix} m_{11} & \cdots & m_{1N} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ m_{N1} & \cdots & m_{NN} \end{pmatrix}\begin{pmatrix} \phi_1^N \\ \cdot \\ \cdot \\ \phi_N^N \end{pmatrix} \equiv M^N\cdot\phi^N. \end{aligned} \tag{14}$$

Setting $\delta\phi_i^N = (\phi_i^N)'' - \phi_i^N$ for all i and subtracting $(M^N)''\cdot\phi^N$ from both sides gives

$$\begin{pmatrix} m_{11} & \cdots & m_{1(N-1)} & \gamma m_{1N} \\ \cdot & \cdots & \cdot & \cdot \\ m_{(N-1)1} & \cdots & m_{(N-1)(N-1)} & \gamma m_{(N-1)N} \\ m_{N1} & \cdots & m_{N(N-1)} & m_{NN} \end{pmatrix}\begin{pmatrix} \delta\phi_1^N \\ \cdot \\ \cdot \\ \delta\phi_N^N \end{pmatrix} = (1-\gamma)\begin{pmatrix} m_{1N} \\ \cdot \\ m_{(N-1)N} \\ 0 \end{pmatrix}\phi_N^N. \tag{15}$$

Solving Eq. (15) gives

$$\begin{aligned}&\begin{vmatrix} m_{11} & \cdots & m_{1(N-1)} & \gamma m_{1N} \\ \cdot & \cdots & \cdot & \cdot \\ m_{(N-1)1} & \cdots & m_{(N-1)(N-1)} & \gamma m_{(N-1)N} \\ m_{N1} & \cdots & m_{N(N-1)} & m_{NN} \end{vmatrix}\delta\phi_i^N \\ &\quad= (1-\gamma)\begin{vmatrix} m_{11} & \cdot & m_{1(i-1)} & m_{1N} & m_{1(i+1)} & \cdot & m_{1(N-1)} \\ m_{21} & \cdot & m_{2(i-1)} & m_{2N} & m_{2(i+1)} & \cdot & m_{2(N-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ m_{(N-1)1} & \cdot & m_{(N-1)(i-1)} & m_{(N-1)N} & m_{(N-1)(i+1)} & \cdot & m_{(N-1)(N-1)} \end{vmatrix}\phi_N^N, \end{aligned} \tag{16}$$

or

$$\begin{aligned}&\left(\gamma\left|M^N\right| + (1-\gamma)\left|M^{N-1}\right|\right)\delta\phi_i^N \\ &\quad= (1-\gamma)\begin{vmatrix} m_{11} & \cdot & m_{1(i-1)} & m_{1N} & m_{1(i+1)} & \cdot & m_{1(N-1)} \\ m_{21} & \cdot & m_{2(i-1)} & m_{2N} & m_{2(i+1)} & \cdot & m_{2(N-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ m_{(N-1)1} & \cdot & m_{(N-1)(i-1)} & m_{(N-1)N} & m_{(N-1)(i+1)} & \cdot & m_{(N-1)(N-1)} \end{vmatrix}\phi_N^N. \end{aligned} \tag{17}$$

Once again, the perturbation $\delta\phi_i^N$ on every scatterer, including the Nth scatterer, is represented using only the original wave field quantities. Eq. (17) has a rather surprising implication. If one could freely control γ (which needs further careful examination in certain cases) and $|M^N| \neq |M^{N-1}|$ (which is usually true), one could always induce resonance in a non-resonant scattering system by setting $\gamma = 1/(1-|M^N|/|M^{N-1}|)$. Since resonance often corresponds to persisting bouncing of wave between scatterers, it's possible to trap wave inside a scatterer field for a significant amount of time by tuning the scattering strength of an arbitrary scatterer, according to Eq. (17). This could be an alternative to the recent experimental work of freezing photons in an atomic cloud [12].

6. Perturbing multiple scatterers

By repeated use of Eq. (4), the removal of a total of m scatterers can be viewed as the sequential removal of a single scatterer at a time, i.e.,

$$\begin{aligned}\phi^{N-m}(\vec{x};\omega)-\phi^{N}(\vec{x};\omega) &= \sum_{j=1}^{m}\bigl(\phi^{N-j}(\vec{x};\omega)-\phi^{N-j+1}(\vec{x};\omega)\bigr)=\sum_{j=1}^{m}\delta\phi^{N-j+1}(\vec{x};\omega)\\ &=\epsilon(\omega)\left\{\sum_{j=1}^{m}\left(\sum_{i=1}^{N-j}G_{\vec{x},i}\delta\phi_i^{N-j+1}-G_{\vec{x},N-j+1}\phi_{N-j+1}^{N-j+1}\right)\right\}\\ &=\epsilon(\omega)\left(\sum_{i=1}^{N-m}G_{\vec{x},i}\sum_{j=1}^{m}\delta\phi_i^{N-j+1}-\sum_{i=N-m+1}^{N}G_{\vec{x},i}\phi_i^{N}\right),\end{aligned} \tag{18}$$

where $\delta\phi_i^{N-j+1}$ are given by equations similar to Eq. (7). In the last form given, all the canceling terms have been removed. Interestingly, the differential contributions coming from the remaining scatterers are summed over the m systems having j scatterers removed, whereas the missing contributions from the removed scatterers rely only on the N-scatterer system. It's the consequence of the linearity of the system, which makes the overall change on any remaining scatterer equal the linear sum of the changes induced in each removal of the m scatterers.

By linearity, the collective change in wave field due to shifting m scatterers is conveniently given as

$$\bigl(\phi^{N}(\vec{x};\omega)\bigr)'-\phi^{N}(\vec{x};\omega)=\epsilon(\omega)\sum_{i=1}^{N}G_{\vec{x},i}\sum_{j=1}^{m}\bigl(\delta\phi_i^{N}\bigr)_j,$$

where $(\delta\phi_i^N)_j$ is the change $\delta\phi_i^N$ in the incoming wave on the ith scatterer due to the jth shifting, given by Eq. (12).

Similarly, when m scatterers change their scattering strength, the collective change in wave field is

$$\bigl(\phi^{N}(\vec{x};\omega)\bigr)''-\phi^{N}(\vec{x};\omega)=\epsilon(\omega)\sum_{i=1}^{N}G_{\vec{x},i}\sum_{k=1}^{m}\bigl(\delta\phi_i^{N}\bigr)_k,$$

where $(\delta\phi_i^N)_k$ is the change $\delta\phi_i^N$ in the incoming wave on the ith scatterer due to the kth scatterer's change in scattering strength, given by Eq. (16). The m scatterers can change their scattering strength by different factors γ. That is, γ in Eq. (16) can be replaced with a different γ_k each time calculating $(\delta\phi_i^N)_k$.

Above are the collective perturbations due to modifying multiple scatterers. They are analytical and strict. At certain weak scattering limit, the various $\delta\phi_i^N$ can be approximated using low scattering terms, therefore offering a convenient analytical study of perturbation effects.

7. Conclusion

In the preceding sections, we have expressed the changes in the overall incoming wave on an arbitrary scatterer in terms of the unperturbed wave field for removing a scatterer, shifting a scatterer, and replacing a scatterer with a different one, respectively, and then given the overall perturbation due to changing multiple scatterers in compact forms. Most of the results can be intuitively understood in the scattering path picture. The equations can be easily solved numerically to compare the changes on different scatterers, which can help reveal the correlation strength between a remaining scatterer and the removed one(s). Also at the weak scattering limit in a ballistic transport system, the few lowest order scattering terms may approximate the change on a scatterer's overall incoming wave field analytically. The formulas are simple and intuitive, should be very helpful to research on perturbations in atomic, mesoscopic and even classical scattering systems.

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