Comment on "Ehrenfest times for classically chaotic systems"

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In a recent Rapid Communication [P. G. Silvestrov and C. W. J. Beenakker, Phys. Rev. E **65**, 035208(R) (2002)], the authors, Silvestrov and Beenakker, introduce a way to lengthen the Ehrenfest time τ for fully chaotic systems. We disagree with several statements made in their paper, and address the following points essential to their conclusions: (1) it is not true that all semiclassical approximations for chaotic systems fail at a so-called "log time" $\tau \propto -\ln(\hbar)$, differing only by a numerical coefficient; and (2) the limitation of the semiclassical approximation as expressed in the authors' Eq. (8) is not limited by their argument leading to Eq. (12).

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It is important to distinguish between the correspondence of quantum and classical dynamical propagations, and the validity of semiclassical approximations. If one takes the Ehrenfest time τ to be the upper limit for which a quantum mechanical wave packet is described by solving classical equations of motion without invoking a semiclassical construction of the wave packet, then the Ehrenfest time increases logarithmically slowly for chaotic systems as $\tau \propto \lambda^{-1}\ln(S/\hbar)$ [2,3]; there is no controversy on this point. In this expression λ is sum of the positive Lyapunov exponents, and S is some characteristic classical action such as that of the shortest periodic orbit. If, instead, one defines τ as the time scale beyond which the time-dependent WKB approximation [4] no longer faithfully reproduces the quantum propagation of a wave packet in full detail, then τ is not a so-called "log time," but is proportional to inverse algebraic powers of $\hbar$ [5–7].

The precise exponent in the breakdown time scale has been shown to depend on a few basic features of the chaotic dynamical system being considered. We mention work on three separate paradigms of chaos. It was shown in the stadium billiard [5] that $\tau \propto \hbar^{-1/2}\ln S/\hbar$ (essentially $\hbar^{-1/2}$). The $\hbar^{-1/2}$ behavior was linked to the fact that the stable and unstable manifolds associated with trajectories in the stadium have discontinuities in their slopes where they fold over upon themselves. The $\ln S/\hbar$ part of the expression is due to the "stickiness" of phase space in the neighborhood of the marginally stable bouncing ball trajectories. In contrast, a general dynamical system possessing stable and unstable manifolds that are continuous in their slopes gives $\tau \propto \hbar^{-1/3}$ [6]; this was illustrated with the kicked rotor. A third example that has been studied extensively is the quantum bakers map. There it was shown that for some quantities, the breakdown time scale could be as great as $\tau \propto \hbar^{-1}$ [7], although $\hbar^{-1/2}$ was typical [8].

Note that the semiclassical approximations in Refs. [5–8] involve no uniformizations or caustic corrections. They are, in fact, either exactly or poor man's versions of the standard WKB method, and developed specifically for chaotic systems. For wave packets, the standard time-dependent WKB method involves sets of complex trajectories [9]. Nevertheless, in the above cited work on τ, no classically nonallowed processes are taken into account. One essential ingredient relied upon in these works, the "area-$\hbar$ rule" is contained in Ref. [3]. This rule is in contradiction with the argument of Ref. [1] leading to Eq. (12), which contains the relation $\hbar^{7/6-c} \ll 1$, c being the coefficient of proportionality in the log time scale relation. The consequences of the area-$\hbar$ rule carefully considered in conjunction with the geometrical properties of evolving stable and unstable manifolds give a precise formulation of the semiclassical breakdown due to caustics and the resultant algebraic time scales [5–7]. The crucial point is that the *distance* between local classical manifolds (the criterion used by Silvestrov and Beenakker) is actually of no importance—what matters is the *area* enclosed by following the manifold from one branch to the next in a given locality. To miss this point unfortunately leads to a qualitatively different and incorrect result.

Finally, we do agree with the authors that there should be important or morphological distinctions in the nature of evolving wave packets as they surpass each relevant time scale. Examples include interference phenomena necessarily arising beyond the log time, and localization effects sometimes supressing classical diffusion beyond algebraic time scales [10].

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