Lab 3. Gauss's Law Tutorial

Goals

- To understand and explain in words the physical meaning of Gauss's law.
- To employ symmetry arguments to determine the direction of the electric field for simple charge distributions.
- To use Gauss's law to calculate the the electric field produced by an appropriate charge distribution.

Introduction

In mathematical form, Gauss's Law can be written as:

$$\oint_S \mathbf{E} \cdot \mathbf{dA} = \frac{q_{Enclosed}}{\varepsilon_0} \tag{3.1}$$

Problem 1

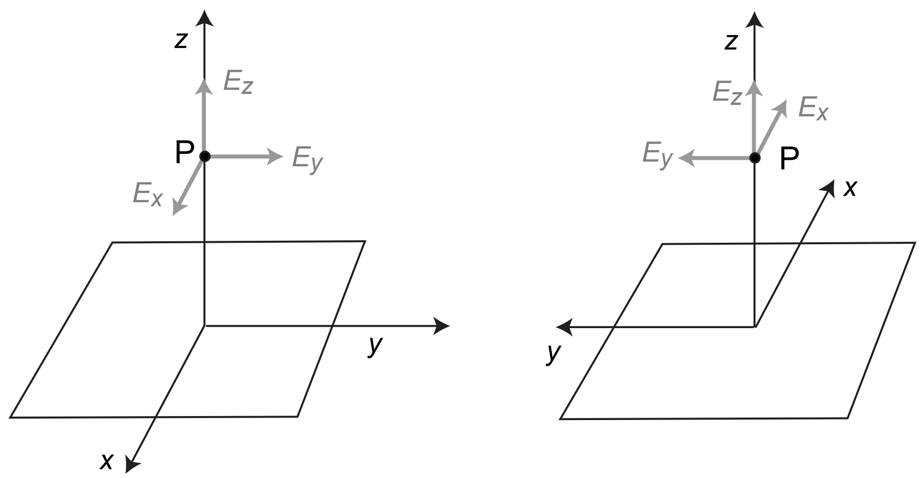
Describe a procedure for applying Gauss's Law of electromagnetism in your own wordsÑ without using equations. Define the following terms carefully:

Surface element	$\mathbf{dA}$
Closed surface	S
Enclosed charge	$q_{Enclosed}$

Describe the dot product in Equation 3.1 in terms of vector components. Describe the process of integration in terms of a sum over many small surface elements. It is a good idea to sketch out your solutions on scratch paper before entering them in your lab notebook.

Using symmetry to determine the direction of the electric field

Gauss's Law can be used to determine the magnitude of the electric field in several important geometries. Gauss's Law is always true, but it doesn't help you determine the magnitude of the electric field unless you know the direction of the field first. In the special cases where Gauss's Law can be used to calculate the electric field, the direction of the field can be determined from the symmetry of the charge distribution. Figure 3.1 shows a uniformly charged, square plate in the x-y plane with its center at the origin. Rotating the charge distribution by 180° around the z-axis does not change the distribution of charge. Every charge element that is moved by rotation is replaced by another, identical charge element after rotation. Therefore, we say that the charge distribution is symmetric with respect to a 180° rotation about the z-axis.



(a) Original charge distribution. (b) Charge distribution after 90° rotation.

Figure 3.1. A thin, uniformly charged square plate in the x-y plane, with its center at the origin of an x-y-z coordinate system before and after rotation by 90°. The gray vectors at Point P show the three components of **E** at Point P. The charge distributions before and after rotation are identical. Therefore we expect identical electric fields. Although the z-component of **E** is the same before and after rotation, the x- and y-components change sign. The only way these observations can be reconciled is if $E_x = E_y = 0$.

Symmetry arguments rely on two principles. First, changing the orientation of the source of an electric field (the charge) changes the orientation of its electric field in the same way. Second, identical charge distributions produce identical fields. Figure 3.1 shows that E_x and E_y at Point P change sign after a 180° rotation about the z-axis. They rotate along with the plate. But this rotation does not change the charge distribution, so it cannot change the electric field. The only way these two facts can both be true is if the x- and y-components of the electric field at Point P are zero. In contrast, E_z is not changed by this rotation. Therefore the z-component of the electric field does not have to be zero. The electric field at Point P must point in the $+z$ or $-z$ directions.

This procedure yields the same result for all points on the z-axis. The procedure fails for points with nonzero x- and y-components, unless the square is infinite in extent. We cannot use this procedure to find the direction of the electric field for points that are not on the z-axis, except for the special case of a plate that extends to infinity in the x- and y-directions.

Three useful groups of symmetry operations are described below.

Translations (straight-line displacements)

Charge distributions with translational symmetry in the x-direction are not changed when the object is moved in the x-direction. A infinite, charged plate perpendicular to the z-axis has translational symmetry in the x- and y-directions. Movements in the x- and y-directions do not change the charge distribution, because every patch of moved charged is replaced by an identical patch of charge from elsewhere on the plate. Since the charge distribution is not changed if you move the plane in the x- or y-directions, the electric field must not be changed by this motion. That is, the electric field vector at any Point P above a charged, infinite plane cannot depend on the x- or y-components of that point.

Rotations

Charge distributions with rotational symmetry are not changed by a rotation about some axis. The axis and the angle of rotation must be specified, although some shapes are symmetric about a special axis for all angles. The square in Figure 3.1 is symmetric under rotations of 90°, 180°, and 270° about the z-axis.

Reflections

The difference between a vector and its reflection is similar to the difference between a vector and its reflection in a mirror. Reflection through the x-y plane changes the sign of vector z-components without changing their magnitude. The x- and y-components are not changed. You will not need to use reflections in this exercise, but they can be useful.

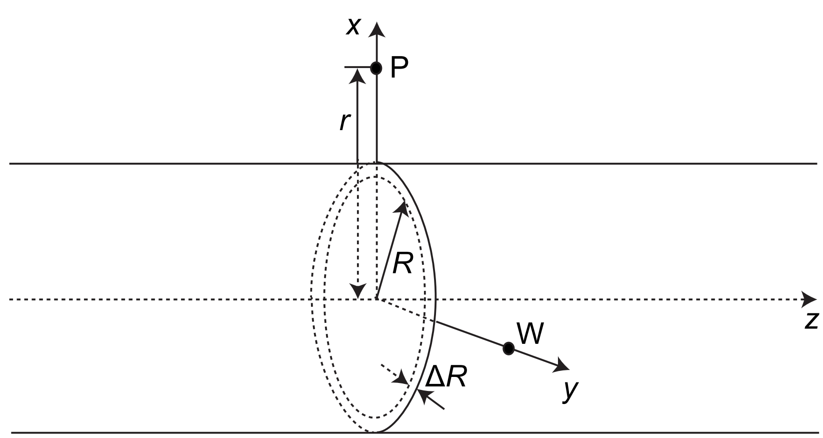


Figure 3.2. Sketch representing an infinitely long cylindrical shell (like a pipe) of radius R and thickness ΔR, centered on the z-axis.

Problem 2

Figure 3.2 represents an infinitely long cylindrical shell of radius R and thickness ΔR, centered on the z-axis. Assume that it is made of a material with a uniform, positive volume charge density ρ.

Sketch "before" and "after" pictures of this charge distribution and the electric field components at the point of interest for each symmetry operation, as in the example above.

(a) Use symmetry arguments to show that the electric field outside the cylinder at Point P on the x-axis is directed in the $+x$-direction. (Rotate the cylinder 180° about the x-axis.)

(b) Use symmetry arguments to show that the electric field outside the cylinder at Point Q on the y-axis is directed in the $+y$-direction.

(c) Use symmetry arguments to show that the electric field is directed along a cylinder radiusÑ that is, pointing directly toward or away from the z-axis. Further, show that the magnitude of the electric field is constant on circles of radius $r = \sqrt{x^2+y^2}$, the distance from the z-axis to Point P.

(d) Use additional symmetry arguments to show that the electric field outside the cylinder does not depend on the z-component of position.

Constructing a Gaussian surface

To perform the integral in Gauss's Law, one must be able to compute the dot product inside the integral:

$$\oint_S \mathbf{E}\cdot\mathbf{dA} = \frac{q_{Enclosed}}{\varepsilon_0} \quad \text{where} \quad \mathbf{E}\cdot\mathbf{dA} = EdA\cos\theta \tag{3.2}$$

and θ is the angle between **E** and **dA**. The circle on the integral sign indicates that the surface must be closed. Like cubes or spheres, closed surfaces are composed of pieces whose orientations vary. Since the direction of **dA** is along the surface normal direction, it changes depending where you are on the closed surface. From Problem 2, we know the direction of **E** due to an infinitely long cylindrical shell. We get to choose the surface, and so have some control over the direction of **dA**. The best we can hope for is to make $\mathbf{E}\cdot\mathbf{dA}$ constant for each part of the surface. This requires that θ be constant for each part of the surface. For closed surfaces, the only workable angles are 0°, 90°, 180° and 270°. Then $\cos\theta = 0$ or ± 1.

Problem 3

Sketch a Gaussian surface for the infinitely long cylindrical shell in Figure 3.2, making sure that the angle between **E** and **dA** is one of the approved angles for each side of the closed surface. Since the surface must be closed, your Gaussian surface cannot be infinitely long. Show the dimensions of the Gaussian surface on your sketch. Write the Gauss's Law integral as the sum of the integrals over each part of your surface, using the fact that $\cos\theta = 0$ or ± 1.

Calculating the flux through the Gaussian surface

Even though we have determined θ for each of the required flux integral(s), we still must be careful of the functional dependence of the remaining scalars E and dA. You can't calculate an integral unless you know the function. This is not a problem where $\cos\theta = 0$, as the integral for those parts must equal zero. But for surfaces where $\cos\theta = \pm 1$, the integral can be a problem. Practically, the only hope for a solution is if the magnitude E is constant. Fortunately, you showed in Problem 2 that for a fixed r, the magnitude of the electric field E is indeed constant. If (and only if) r is constant for the parts of your Gaussian surface where $\cos\theta = \pm 1$, you can perform the integral. After factoring out the constant factor of E, the remaining integral should have the form:

$$\int dA = A \tag{3.3}$$

where A is the area of a part of your Gaussian surface where $\cos\theta = +1$ or -1.

Problem 4

Calculate the total flux through your Gaussian surface S.

Calculating the charge enclosed by the Gaussian surface

Since S is a closed surface, with a definite inside and outside, it encloses a well defined volume. If all the charges in the system are simple point charges, one can simply identify which point charges are inside the volume and sum their values. Another simple case is when the charge density in the volume is uniform, or constant. Then the enclosed charge is given by the product of the volume V inside S and the charge density ρ; that is, $q_{Enclosed} = \rho V$. Care must be taken to include only the charge inside S. If part of a charge distribution is not inside S (that is, some parts poke through the surface), only the part inside S contributes to $q_{Enclosed}$.

If the charge density is a function of R only, it can still have rotational symmetry. (In this case, the shape is not changed by any rotation about the axis of symmetry.) Then the enclosed charge may be found by integration. To minimize confusion, we will use the variable R to refer to the radial coordinate of a position in the charge distribution (the cylinder). We will use the variable r to refer to the radius of our Gaussian surface.

In your calculus class, you used the method of cylindrical shells to determine the volume of shapes with rotational symmetry. You can use the same method to determine the total charge in such an object by introducing a factor of ρ, the volume charge density. In the shell method, the volume of a thin cylindrical shell is given by[1]

$$\Delta V = 2\pi h R dr \tag{3.4}$$

where h is the (constant) length of the shell. To see that this must be true, consider a solid cylinder with radius R_{Cyl} whose charge density might be a function of R. In this case one can approximate

[1] See, for example, the current calculus text: William Briggs, Lyle Cochran, and Bernard Gillet, *CalculusÑEarly Transcendentals* (Pearson, Boston, 2015), Section 6.4, on the shell method. Note that Briggs treats *much* more complicated shapes than we do. We have a simple cylinder with a fixed height. We will eventually treat the case of variable charge density. The charge density, however, will be constant inside every thin shell of radius R. That is all we need.

the volume of the cylinder as the sum of the volumes of a series of N thin cylindrical shells of radii $R_1, R_2, R_3 ... R_N$. If we take the thickness of each shell to be $\Delta R = R_{Cyl}/N$, we can construct a series of shells with radii $R_J = J\Delta R$, where $(J = 1, 2, 3...N)$. As N goes to infinity the sum of the shell volumes V_J becomes an integral, and the integral yields the exact value of V_{Cyl}. (This is the definition of an integral according to Riemann.) The progression from thin shells to integrals can be written:

$$V_{Cyl} = \lim_{x\to\infty} \sum_{J=0}^{N} \Delta V_J = \lim_{x\to\infty} \sum_{J=0}^{N} 2\pi h R_J \Delta R = \int_0^{R_{Cyl}} 2\pi h R dR \tag{3.5}$$

To find the charge enclosed in the entire cylinder, q_{Cyl}, one need only add a factor of ρ to the integral.

$$q_{Cyl} = \int_0^{R_{Cyl}} 2\pi \rho h R dR \tag{3.6}$$

You can find q_{Cyl} for almost any charge distribution $\rho(R)$ that depends only on R. If the radius of your Gaussian surface is greater than the radius of the cylinder, $q_{Enclosed} = q_{Cyl}$; the upper limit of integration is then R_{Cyl}, as in 3.6. If the radius of your Gaussian surface is less than the radius of the cylinder, you must include only the charge inside the Gaussian surface. To get $q_{Enclosed}$, you reduce the upper limit of the integral from R_{Cyl} to r, the radius of you Gaussian surface.

Problem 5

Compute the total charge inside in a cylinder of length h and radius R_{Cyl} when $\rho(R) = \alpha R$. Use the result to compute the electric field produced by the cylinder at points outside the cylinder $(r > R_{Cyl})$. Note that since $r > R_{Cyl}$, the Gaussian surface (with radius r) encloses all the charge in the cylinder. State the direction of the electric field inside and outside the cylinder when $\alpha > 0$, that is, when the cylinder carries positive charge.

Problem 6

Find the charge enclosed by a Gaussian surface as a function of its radius, r, when $\rho(R) = \alpha R$, for the case of $r < R_{Cyl}$. Since $r < R_{Cyl}$l, a Gaussian surface with radius r encloses only part of the cylinder's charge. Use the result with the rest of Gauss's Law to compute the magnitude of the electric field inside the cylinder as a function r for $r < R_{Cyl}$.

Before you leave the lab please:

Turn your work in to your teaching assistant.

Work done at home will receive no more than 50% credit.