Lab 12. Spring-Mass Oscillations

Goals

- To determine experimentally whether the supplied spring obeys Hooke's law, and if so, to calculate its spring constant.
- To find a solution to the differential equation for displacement that results from applying Newton's laws to a simple spring-mass system, and to compare the functional form of this solution to the experimental oscillation you observe.
- To determine the spring constant by another method, namely, by observing how the oscillation frequency changes as the mass hanging on the end of the spring is varied.
- To learn how to use the Òuser-defined fitÓ capability of the DataStudio software to fit more complicated experimental data.
- To add additional air resistance to the oscillating system and compare the resulting displacement as a function of time with the theoretical prediction given.

Introduction

If you hang a mass from the bottom end of a spring, then pull the mass down and release it, the mass will oscillate up and down. In this experiment we explore the nature of the force exerted by a "real" spring when it stretches. We determine if the resulting force oscillation is "simple harmonic" and examine the effect of energy loss on its motion. The term "simple harmonic" is applied to oscillatory motion that can be characterized by a sinusoidal function; that is, the displacement follows a simple sine or cosine function.

This lab makes extensive use of curve fitting routines, where the computer fits a model to your experimental data. Normally a model that omits important features of the experiment will fail to describe the data well. Unfortunately, making a model more complex can give it the ability to fit data for which the theory does not apply. (Some models can "fit an elephant"!) One defense against this is to examine any predictions of the model (or theory) for reasonableness.

Force exerted by a stretched spring

In this exercise, you are to plot the force exerted by a spring as a function of its "stretch" (not the overall length). Suspend the spring from a force sensor. Start by adding a 50-g mass to the mass hanger, which also has a mass of 50 g, to make a total of 100 g of mass hanging from the spring. Then increase the hanging mass in 100 g increments up to a total of 1200 g. Devise a method to measure the stretch of the spring. Then take the data. Remember to zero the force sensor appropriately and to use SI units.

From your graph determine a mathematical equation relating the spring force to the stretch of the spring. Compare your result to the Hooke's Law model described in your textbook. Can you characterize your real spring with a unique value of the spring constant (sometimes called the force constant) as in Hooke's Law? Based on your data and analysis, how "ideal" is your spring?

Spring-mass oscillations (neglect damping)

Applying Newton's Second Law to a mass hanging on a ÒmasslessÓ spring that can be modeled by Hooke's Law, one finds that:

$$M\frac{d^2x}{d^2t} = -kx \quad \text{that is,} \quad ma = F \tag{12.1}$$

where m is the mass hanging on the spring and x is the distance the spring has been stretched from its equilibrium position with the mass hanging at rest.

What does the function $x(t)$ that satisfies Equation 12.1 look like? Guessing happens to be a "tried and true" technique for solving so-called differential equations. What common function do we know whose second derivative gives us the original function back again except with a minus sign? Hint: Try a function like $x(t) = A\cos(Gt)$. Under what conditions does it satisfy the equality of Equation 12.1?

From your knowledge of oscillatory motion you should recognize that G corresponds to the angular frequency of oscillation (often represented by the Greek letter ω). And of course, the angular frequency in radians/sec is just 2π times the ordinary frequency in Hz.

Experiment set-up

Hopefully you are convinced you that the spring you are using is well described by Hooke's Law. Since the spring force is directly proportional to the displacement from equilibrium, Equation 12.1 implies that both the displacement and the spring force should be sinusoidal for the oscillating mass.

Hang a 1-kg mass from the spring and set up the force sensor to measure the force oscillations. Increase the sampling rate of the force sensor to 20 Hz. Zero the force sensor when the mass is hanging at equilibrium. Displace the mass about 5 cm from its rest position and release it. Using DataStudio display the force on a graph for an elapsed time of 10 or 15 seconds. Now find "fit" in the toolbar at the top of the graph window and choose Sine Fit. This fits a sinusoidal function of the form $A\sin[2\pi(t-C)/B]+D$ to your data and displays the "best-fit" values of the constants A, B, C, and D as found by a least squares technique. Comment on how accurately the fitted sinusoidal function matches the actual data. That is, does this function describe your data? The value of the period of the oscillation is given by the constant B. Check to make sure that the value of B reported by the Sine Fit function is reasonable. For instance, you can estimate the period using a watch or the computer clock over several periods. When a curve fitting routine makes an error, it is usually a large one. You do not need a very precise period measurement to perform a useful check on the results of the curve fitting routine.

Perform a simple test of the effect of amplitude on the period of oscillation by displacing the mass about 20 cm from its equilibrium position and releasing it. Display the force on a graph and try the Sine Fit function on this data. Again comment on how accurately the fitted function matches the data and note the value of the period of oscillation. Based on this limited data, how does the oscillation period depend on the amplitude of the oscillation? Compare this behavior with the effect of amplitude on the period of the simple pendulum, which you measured in an earlier lab.

Effect of mass on the oscillation period

Vary the total hanging mass including the mass of the 50-g hanger from 200 g to 1200 g, determining the period of the oscillation for each mass value using the techniques from the previous section. It is a good idea to zero the force sensor with the mass at equilibrium prior to each run. For an ideal spring, the angular frequency, ω, of an oscillating spring-mass system is related to the spring constant, k, and the hanging mass, m, by the relation:

$$\omega = \left(\frac{k}{m}\right)^{1/2} \tag{12.2}$$

We hope to determine k by measuring the period T as a function of the mass m on the end of the spring. Because the slope of a line can be determined with low uncertainties, we want the modify Equation 12.2 to get the equation of a line with slope k. Rewrite ω in terms of the period, T, and solve Equation 12.2 for the mass, m. Show/explain why making a graph of m (on the vertical axis) as a function of $T^2/4\pi^2$ (on the horizontal axis) should be a straight line with a slope value equal to the spring constant, k, and a vertical axis intercept of zero. Make such a graph with your data. Compare the slope value of your graph with the spring constant determined using forces measurements. Do they agree within the uncertainty limits of each? Does your graph have a zero intercept as predicted by the analysis of an ideal spring? What is the intercept value and corresponding uncertainty? What are the units of the intercept value?

Hooke's Law applies to "ideal", that is, massless springs. For ideal springs, the oscillation period goes to zero as the hanging mass is reduced to zero. Thus the oscillation frequency would approach infinity as the hanging mass approaches zero. Remove the hanging mass from your spring, stretch it a small amount (0.5–1.0 cm), and let it go. Is the oscillation frequency of your spring by itself infinite? How does it differ from an ideal spring?

When m (on the vertical axis) is plotted as a function of $T^2/4\pi^2$ (on the horizontal axis), the intercept value will be negative with a magnitude known as the "effective mass of the spring." Using energy considerations and some simplifying assumptions, one can show that the effective mass should be about one-third of the total mass of the spring. How closely does the magnitude of your intercept value compare to one-third of your spring mass?

Spring-mass oscillations with damping

The term "damping" is just a short way of saying that there are frictional forces which convert the mechanical energy (potential and kinetic) of a system into heat. In our case we will use a thin piece of cardboard moving back and forth through the air to provide damping. The cardboard piece has a small hole punched in the middle allowing it to slide over the top of the mass hanger. Then you can place any additional mass on top of the cardboard piece to hold it firmly in place. For this part of the experiment we wish to have a total of 500 $\pm$ 5 g (including the mass of the cardboard and the 50-g hanger; record the actual value used) hanging on the spring.

With the hanging mass and cardboard in place stretch the spring downward 8–10 cm from equilibrium and release it. Use DataStudio to plot the force as a function of time for 40–60 s. Make sure that force sensor reads zero at equilibrium, since we will assume this to be case when we fit Equation 12.3 below to our data. You will notice that the amplitude of the oscillation decreases significantly during the experiment due to the damping effect of air on the piece of cardboard.

As discussed in your textbook, when the damping force is relatively small and proportional to the velocity of the object, the oscillations can be described by a sinusoidal function with an amplitude that decreases exponentially (that is a negative power of e). We can check whether this is true for the present system by fitting such a function to our data and determining whether it fits appropriately.

At the top of the graph window, click on "Fit" and choose "User-Defined Fit" from the drop-down menu. None of the other options is appropriate for our case. A small box will appear in the graph window saying that the "User-Defined Fit" could not fit the data. Of course this is the case since we haven't specified the functional form of the fitting function to be used. Double click inside the small box to bring up another window that allows us to specify the function to use in the fitting procedure. In the space provided type:

$$a * \exp(-b * x) * \sin(6.283 * (x - c)/d) \tag{12.3}$$

This tells the software to attempt fitting a function of the form $ae^{-bt}\sin[2\pi(t-c)/d]$ to your data, where DataStudio represents the time t by the variable "x". The value of "d" is the period of the oscillation. The value of "a" is the amplitude of the oscillation at time $t = 0$, "b" is related to the damping of the system, and "c" adjusts the phase of the sine wave to accommodate positive or negative values of the function at time $t = 0$.

You are also asked to give some initial values to these parameters. Try a = 2.0, b = 0.03, $c = 0$, and d = 1.0. After you have specified the functional form of the fitting function and have assigned initial values to the fitting constants, click on "Accept" at the upper right hand side of the small window. The fitting may take up to a minute to complete because the number of data points is large (about 1000). When the fit is done, the actual "best-fit" values of the four constants will be shown in the fit window to the left of your initial guesses and also in a small box within the graph window. Uncertainties (in the form of standard errors) will also be displayed for each value. In the graph window check to make sure that the fitted function actually does a good job of fitting the data. If the amplitude or phase of the fitted function differs significantly from the data (check this carefully!), then you will need to choose some other initial values of the constants and try again.

How closely does the fitted curve match your actual data? You may wish to expand the time scale of a portion of your data so the fitted curve and the actual data can be seen more clearly and print it out to support your conclusions here. What would happen if you tried to fit this data with the Sine Fit function. Try it!

[The software starts with your initial guesses for each constant and calculates the value of the function at each time a data point was acquired. Then it takes the difference between the calculated "y- value" (force in this case) and the measured "y-value" for the data point, and sums the squares of these differences for all the data points. The program then minimizes the sum of the squares (hence the name Òmethod of least squaresÓ) by guessing slightly different values for the constants and recalculating the sum of the squared differences. When it is satisfied that an actual minimum in the sum has been found, the program displays the values of the constants that give that minimum value.]

The decay constant b above describes how rapidly the amplitude of the oscillation falls with time. It is proportional to the "damping constant", which describes the effect of air resistance on the motion. The reciprocal of b has units of time, and indicates the time required for the amplitude to drop by a factor of $1/e$ (about 1/3) of its initial value. Estimate the time required for the oscillation amplitude to reach $1/e$ of its initial amplitude using a watch or the computer clock. Is the value of b reported by the User Defined Fit consistent with your estimate? Again, a precise estimate is not required.

Compare the period of the damped pendulum with the period without damping, but with the same total mass ($\pm$ 5 g) on the end of the spring. Use your uncertainties in making this comparison. Explain any differences you see. You may want compare the oscillation frequencies with and without damping using the relationship for damped oscillations given in your textbook. To do this, you will have to covert the b you measured using Equation 12.3 to the b used in the text.

Summary

Summarize your findings.

Curve fitting routines are powerful tools in science and engineering. They are simple examples of computer models or simulations. Normally one derives a model from theoretical considerations, then tests it against experiment. If the model is missing important features, it will generally fail to describe the data well. Unfortunately, making a model more complex can give it the ability to fit data for which the theory does not apply. The best defense against this is usually to examine predictions of the model for reasonableness, as you did above when you checked to see if the output parameters ω and b were reasonable.

Before you leave the lab please:
- Quit DataStudio.
- Straighten up your lab station.
- Report any problems or suggest improvements to your TA.