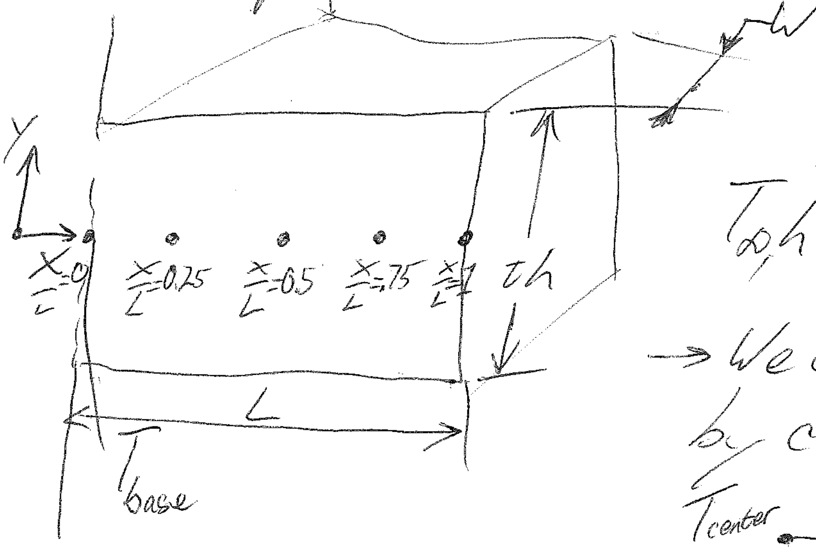
Analytical Solutions to Extended Surfaces / ME576 Sp 2013 4.1

→ Fin approximation
→ Derivation
  → exp vs. sinh & cosh
~~Fin efficiency~~
  → Bessel functions
→ Fin efficiency

→ Thus far all of our problems have truly been 1-D. There are many situations where the temperature distribution is actually 2-D or 3-D but can be approximated as 1-D with little error.

→ This simplification is referred to as the <u>extended surface approximation</u>.



→ Can this problem be solved with a 1-D fin approximation?

☆ Show 2-D temp distribution

→ We can understand this by comparing thermal resistances:

T_{center} — $R_{cond,y}$ — T_s — R_{conv} — T_∞

$$R_{cond,x} = \frac{\Delta x}{kA_c} \approx \frac{L}{k\,th\,W} \qquad R_{cond,y} \approx \frac{th}{4kWL} \qquad R_{conv} \approx \frac{1}{h\,2WL}$$

note error in book

What if $\frac{R_{cond,y}}{R_{conv}} \ll 1$? What happens to temp gradient in y? → 0

☆ show surface approximation plot

When $\frac{\Delta T_{cond,y}}{\Delta T_{conv}} \ll 1$ we can apply the extended surface approximation because temperature is primarily a function of x $T(x)$

$$\frac{R_{cond,y}}{R_{conv}} = \text{Biot number} = \frac{th}{4kWL}\,\frac{2hWL}{1} = \boxed{\frac{th\,h}{2k}}$$

for this situation!!!

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→ If $B_i \ll 1$ then extended surface approximation is valid
→ thin conductive member with a small convective HT coefficient

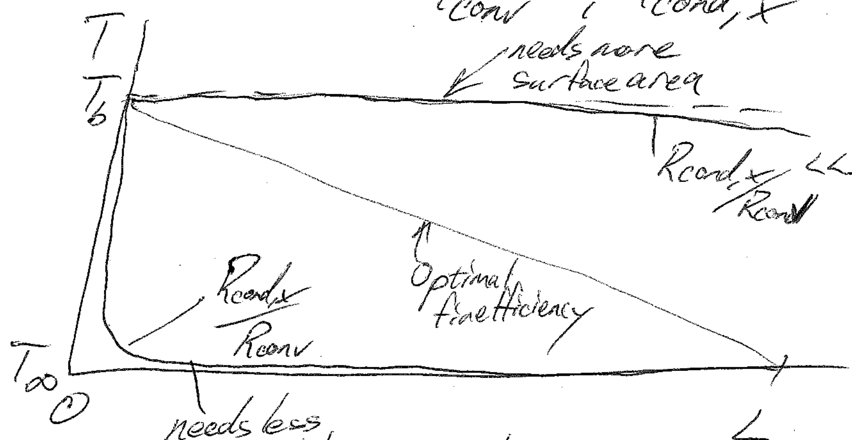
In general $B_i = \dfrac{\text{resistance to conduction in direction to "remove"}}{\text{resistance from surface}}$

or $B_i = \dfrac{\text{resistance you want to neglect}}{\text{resistance you want to consider}}$

→ If we assume extended surface approximation applies then we can anticipate a solution based on R_{conv} & $R_{cond,x}$

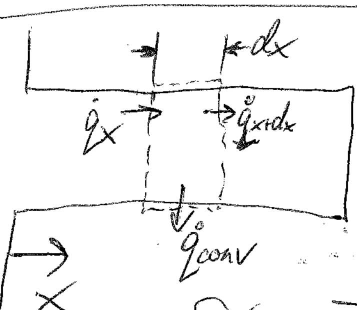
If $\dfrac{R_{cond,x}}{R_{conv}} \ll 1$ What does T gradient look like?

If $\dfrac{R_{cond,x}}{R_{conv}} \gg 1$



Optimal is probably closer to $\dfrac{R_{cond,x}}{R_{conv}} \sim 1$ ← this is likely an efficient fin
→ We'll come back to this.

Derive Analytical Solution:



Step 1: Define Control volume (only differential in 1-D for fin)
→ extends across entire thickness (no ΔT in y)

Step 2: Define energy terms
IN = OUT + STORED

Step 3: Energy Balance: $\dot{q}_x = \dot{q}_{conv} + \dot{q}_{x+dx}$

Step 4: Expand terms & take limit as dx approaches zero: $\cancel{\dot{q}_x} = \dot{q}_{conv} + \cancel{\dot{q}_x} + \dfrac{d\dot{q}}{dx}dx$

~~Change~~ rate of conduction drops due to convection $0 = \dot{q}_{conv} + \dfrac{d\dot{q}}{dx}dx$

Step 5: Substitute rate equations

$$\mathring{q} = -k\,Wth\,\frac{dT}{dx} \qquad \mathring{q}_{conv} = \bar{h}\,\underbrace{per\,d_x}_{\text{Surface area within CV}}\,(T - T_\infty)$$

$$\Rightarrow 0 = \bar{h}\,per\,d_x\,(T - T_\infty) + \frac{d}{dx}\left[-k\,Wth\,\frac{dT}{dx}\right]dx$$

rearrange:

$$\cancel{\frac{d^2T}{dx^2} - \frac{per\,\bar{h}}{kWth}} \quad \frac{-k\,Wth\,\frac{d^2T}{dx}\,dx + \bar{h}\,per\,d_x\,T - \bar{h}\,per\,dx\,T_\infty = 0}{-k\,Wth\,dx}$$

$$\Rightarrow \frac{d^2T}{dx} - \frac{\bar{h}\,per}{kWth}\,T = -\frac{\bar{h}\,per}{kWth}\,T_\infty$$

→ 2nd Order
→ linear
→ non-homogeneous
→ ordinary diff. eq.

Step 6: ~~Solve the ODE~~ $\frac{d}{dr}\left[-kr\frac{dT}{dr}\right] = -\dot{g}\cdots$

→ Last time we were able to multiply both sides by dr to seperate it, this one is not seperable. To solve this type of ODE we need to split it into a homogeneous (T_h) & a particular (T_p) solution: $T = T_h + T_p$

Substitute into ODE: $$\frac{d^2(T_h + T_p)}{dx^2} - \frac{per\,\bar{h}}{kWth}(T_h + T_p) = -\frac{per\,\bar{h}}{kWth}\,T_\infty$$

$$\Rightarrow \underbrace{\frac{d^2T_h}{dx^2} - \frac{per\,\bar{h}}{kWth}\,T_h}_{=0 \text{ for homogeneous differential Equation}} + \underbrace{\frac{d^2T_p}{dx^2} - \frac{per\,\bar{h}}{kWth}\,T_p = -\frac{per\,\bar{h}}{kWth}\,T_\infty}_{\text{Whatever is left over must be the particular differential equation}}$$

Now we will solve the two parts seperately.

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Particular Solution: $\frac{d^2 T_p}{dx^2} - \frac{per\bar{h}}{kWth} T_p = -\frac{per\bar{h}}{kWth} T_\infty$ guess a solution, any one works $T_p = \text{constant}$

$$\frac{d^2(\text{Constant})}{dx^2} - \frac{per\bar{h}}{kWth}\text{Constant} = -\frac{per\bar{h}}{kWth} T_\infty \Rightarrow T_\infty = \text{Constant} = T_p$$

so $\underline{T_\infty = T_p}$

Homogeneous Solution: $\frac{d^2 T_H}{dx^2} - \frac{per\bar{h}}{kWth} T_h = 0$

As engineers, it's not so important that we "solve" the ODE as much as identify which class of functions were developed to solve this homogeneous ODE (e.g. exponential, sine, hyperbolic sin, Bessel, kelvin... etc.)

→ these are just names given to series that solve certain classes of ODEs.

→ In this case: $T_h = C\exp(mx)$ will solve, substitute in

$$\frac{d^2(C\exp(mx))}{dx^2} - \frac{per\bar{h}}{kWth} C\exp(mx) = 0 \Rightarrow Cm^2\exp(mx) - \frac{per\bar{h}}{kWth} C\exp(mx) = 0$$

so $m^2 = \frac{per\bar{h}}{kWth}$ or $m^2 = \frac{per\bar{h}}{kA_c}$ ← this quadratic equation has 2 solutions corresponding to the 2 roots:

or $m = \sqrt{\frac{per\bar{h}}{kA_c}}$ $T_{h,1} = C_1\exp(mx)$ & $T_{H,2} = C_2\exp(-mx)$

The sum of linear ODE solutions is also a solution

$$T_h = T_{h,1} + T_{h,2} = C_1\exp(mx) + C_2\exp(-mx)$$

So the General Solution is $\Rightarrow$ $\underline{T = T_h + T_p = C_1\exp(mx) + C_2\exp(-mx) + T_\infty}$