What if our problem is not semi-infinite? Initially our problem behaves semi-infinite if the thermal wave collapses onto $T(x, t_{1/2})$ in the figure. However, at some point the thermal wave will reach the other side of the computational domain & the overall temp will continue to rise towards $T_s$ ($t \geq t_q$). Self-similar solutions are no longer possible, hence separation of variables (SOV) can be used for analytical solution & the process is similar for 1-D transients & 2-D steady state.

Example:

\[
\begin{align*}
L &= 50 \text{cm} \\
T_{\text{ini}} &= 100k \\
T_s &= 200k \\
k &= 10 \text{ W/m-k} \\
\alpha &= 5 \times 10^{-4} \text{ m}^2/\text{s} \\
\rho &= 700 \text{ kg/m}^3 \\
C_p &= 500 \text{ J/kg-k}
\end{align*}
\]

Initial condition: $T(x, 0) = T_{\text{ini}}$, BCs: $\frac{dT}{dx} = 0
t = 0, L = x_0$

Does this satisfy requirements for SOV?

1) PDE is linear & homogeneous (✓)
2) Boundary is simple (✓)
3) BCs in 1 direction (we only have 1-directional) are linear & HC?

Note: Convective BC @ $x = L$ is not HC, note that initial condition does not have to be HC.

Transform problem: $Q = T - T_0$

PDE: $\alpha \frac{d^2Q}{dx^2} = \frac{dT}{dt}$ IC: $Q = T_{\text{ini}} - T_0$ BCs: $\frac{dQ}{dx} = 0 ~ x = 0$, $-k \frac{dQ}{dx} = h (T_0 - T_s) ~ x = L$

Now the steps are the same as for 2-D SS problems.
Separate the Variables: Assume $u(x,t)$ can be written as the product of a function only of $x$ and a function only of $t$: 

$$u(x,t) = \Phi(x) \Psi(t)$$

Substitute into PDE:

$$\alpha \frac{\partial^2 \Phi}{\partial x^2} = \frac{\partial \Phi}{\partial t} \Rightarrow \alpha \Phi \frac{\partial^2 \Psi}{\partial t^2} = \frac{\partial \Phi}{\partial t}$$

Divide through by $\Phi$:

$$\frac{\partial^2 \Psi}{\partial t^2} = \frac{1}{\alpha \Phi} \frac{\partial \Phi}{\partial t}$$

In order to satisfy this equation at all $x \neq t$, both sides must be equal to a constant.

$$\frac{\partial^2 \Psi}{\partial t^2} = \frac{1}{\alpha \Phi} \frac{\partial \Phi}{\partial t}$$

What difference does sign make? Why we choose $-\Psi$? Set of eigenfunctions must be in $x$.

$$\frac{d^2 \Phi}{dx^2} + \lambda \Phi = 0$$

Solve the eigenproblem: $\Phi = C_1 \sin(2x) + C_2 \cos(2x)$

apply $BC: \Phi(x) = 0$:

$$\frac{d\Phi}{dx} \bigg|_{x=0} = 0 \Rightarrow \frac{d\Phi}{dx} \bigg|_{x=0} = 0 = C_2 \sin(20) - C_2 \cos(20) \Rightarrow C_2 = 0$$

apply $BC: \Phi(x = L)$:

$$-\frac{d\Phi}{dx} \bigg|_{x=L} = \Phi \bigg|_{x=L} \Rightarrow -\frac{dC_2}{dx} \bigg|_{x=L} = C_1 \cos(2L) \Rightarrow -\frac{d\Phi}{dx} \bigg|_{x=L} = C_1 \cos(2L)$$

Substitute in $\Phi$:

$$\Rightarrow \frac{1}{k^2} = \frac{\sin(2L)}{\cos(2L)} \Rightarrow \tan(2L) = \frac{\sin(2L)}{\cos(2L)}$$

$$\Rightarrow k^2 = \frac{\sin(2L)}{\cos(2L)}$$

This is the eigencondition for the problem with an infinite # of eigenvalues.

$$\tan(2L) = \frac{B_i}{k}$$

where $B_i = \frac{R_i}{L}$, we can identify by plotting residual: $\frac{R_i}{L} = \tan(2L)$
Setup range & guess values for each eigenvalue
\[ N_{ee} = 10 \]

\[ \text{Duplicate } i = 1 \text{, } N_{ee} \]
\[ \text{Lower limit } \lambda_i = (i-1) \frac{\alpha}{2} \]
\[ \text{Upper limit } \lambda_i = (i-1) \frac{\alpha}{2} + \frac{\alpha}{4} \]
\[ \text{guess } \lambda_i = (i-1) \frac{\alpha}{2} + \frac{\alpha}{4} \]
\[ \text{end} \]

Implement eigen condition

\[ \text{Duplicate } i = 1 \text{, } N_{ee} \]
\[ \cos(\lambda_i X) = \frac{B_i}{\lambda_i X} \]
\[ \lambda_i X = \cos(\lambda_i X) \]
\[ \text{end} \]

Now solve the non-homogeneous problem for each eigenvalue
\[ \frac{d^2 \theta_i}{dt^2} + \lambda_i^2 \theta_i = 0 \Rightarrow \theta_i = C_{\theta_i} \exp(-\lambda_i^2 x t) \]

\[ \Rightarrow B_i \theta_i \theta_i = C_{\theta_i} \cos(\lambda_i x) C_{\theta_i} \exp(-\lambda_i^2 x t) \Rightarrow \]

\[ \Rightarrow B_i \theta_i = C_{\theta_i} \cos(\lambda_i x) \exp(-\lambda_i^2 x t) \]

Apply IC: \[ \theta_i \big|_{t=0} = T_i \Rightarrow \theta_i \big|_{t=0} = T_i \]

Use orthogonality of eigenfunctions:
\[ \sum_{i=1}^{N_{ee}} C_i \cos(\lambda_i x) \cos(\lambda_i x) = (T_i - T_0) \cos(\lambda_i x) \]
\[
C_i \int_0^l \cos^2(\alpha x) \, dx = (T_{ini} - T_0) \int_0^l \cos(\alpha x) \, dx \implies \\
\frac{C_i}{2} \frac{\cos(\alpha l) \sin(\alpha l) + 2 \alpha l}{\alpha l} = (T_{ini} - T_0) \frac{\sin(\alpha l)}{\alpha l} \\
C_i = \frac{2(T_{ini} - T_0) \sin(\alpha l) \exp(-\alpha x)}{\cos(\alpha l) \sin(\alpha l) + 2 \alpha l} \\
\]

=> Show solution in EES plot

Fourier # \( F_0 = \frac{\alpha t}{L^2} \)

\( F_0 < 0.2 \) thermal wave has not hit wall & semi-infinite solution could apply

\( F_0 > 0.2 \) exact solution can be approximated by single term

\( \text{show plot Pg. 417} \)

\( \text{if } \alpha \ll 0.1 \) exact solution corresponds to lumped capacitance solution