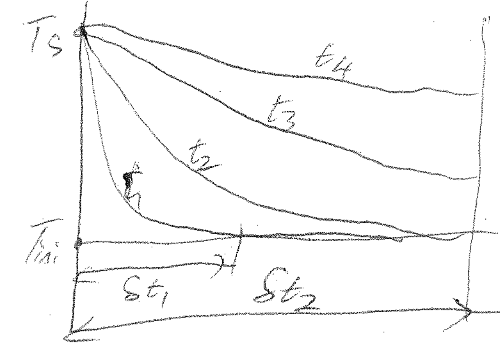
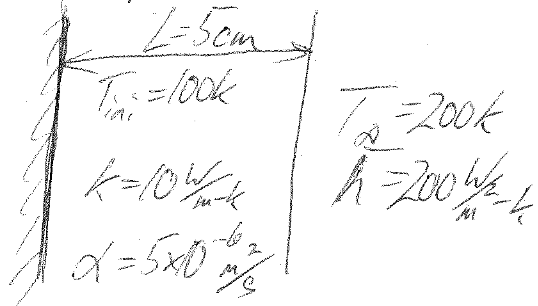
→ What if our problem is not semi-infinite?



→ Initially our problem behaves semi-infinite & the thermal wave collapses onto $\frac{x}{\delta t}$ (t_1 & t_2) in the figure. However, @ some point the thermal wave will reach the other side of the computational domain & the overall temp will continue to rise towards T_s (t_3 & t_4). Self-similar solutions are no longer possible, however Separation of variables (SOV) can be used for analytical solution & the process is similar for 1-D transients & 2-D steady state.

Example:



PDE is same as earlier $\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2} = 0$

Initial Condition: $T_{t=0} = T_{ini}$, BCs: $\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$

$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h\left(T_{x=L} - T_\infty\right)$

Does this satisfy requirements for SOV?

1) PDE is linear & homogeneous ✓
2) Boundary is simple ✓
3) BCs in 1 direction (we only have 1 direction) are linear & HG?

convective BC @ $x=L$ is not HG, note that initial condition does not have to be HG.

→ Transform problem: $\Theta = T - T_\infty$

PDE: $\alpha \frac{\partial^2 \Theta}{\partial x^2} = \frac{\partial \Theta}{\partial t}$ IC: $\Theta_{t=0} = T_{ini} - T_\infty$ BCs: $\left.\frac{\partial \Theta}{\partial x}\right|_{x=0} = 0$

$-k\left.\frac{\partial \Theta}{\partial x}\right|_{x=L} = h\Theta_{x=L}$

→ Now the steps are the same as for 2-D SS problems

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Separate the Variables: Assume $\theta(x,t)$ can be written as the product of a function only of x and a function only of t:

$$\theta(x,t)=\theta_x(x)\theta_t(t)$$

Substitute into PDE: $\alpha\dfrac{\partial^2\theta}{\partial x^2}=\dfrac{\partial\theta}{\partial t}\Rightarrow\alpha\theta_t\dfrac{\partial^2\theta_x}{\partial x^2}=\theta_x\dfrac{d\theta_t}{dt}$

Divide through by $\theta_x\theta_t$: $-\lambda^2=\dfrac{\frac{d^2\theta_x}{dx^2}}{\theta_x}=\dfrac{\frac{d\theta_t}{dt}}{\alpha\theta_t}$ In order to satisfy this equation at all x & t, both sides must be equal to a constant.

What difference does sign make & why we I choose $-$? Set of eigenfunctions must be in x.

So $\boxed{\dfrac{d^2\theta_x}{dx^2}+\lambda^2\theta_x=0}$ & $\dfrac{d\theta_t}{dt}+\lambda^2\alpha\theta_t=0$

Solve the eigenproblem $\theta_x=C_1\sin(\lambda x)+C_2\cos(\lambda x)$ apply BC @ $x=0$:

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0}=0\Rightarrow\left.\frac{d\theta_x}{dx}\right|_{x=0}=0=C_1\lambda\underbrace{\cos(\lambda 0)}_{1}-C_2\lambda\underbrace{\sin(\lambda 0)}_{0}\Rightarrow C_1\lambda=0$$

$$\underline{\theta_x=C_2\cos(\lambda x)}$$

apply BC @ $x=L$:

$$-k\left.\frac{\partial\theta}{\partial x}\right|_{x=L}=\bar{h}\theta_{x=L}\Rightarrow-k\theta_t\left.\frac{d\theta_x}{dx}\right|_{x=L}=\bar{h}\theta_t\theta_{x,x=L}\Rightarrow-k\left.\frac{d\theta_x}{dx}\right|_{x=L}=\bar{h}\theta_{x,x=L}\Rightarrow$$

substitute in θ_x

$$\Rightarrow-k\frac{d}{dx}\left[C_2\cos(\lambda L)\right]=\bar{h}C_2\cos(\lambda L)\Rightarrow-kC_2\lambda\sin(\lambda L)=\bar{h}C_2\cos(\lambda L)$$

$$\Rightarrow\underline{\frac{\bar{h}}{k\lambda}=\frac{\sin(\lambda L)}{\cos(\lambda L)}}$$ this is the eigencondition for the problem with an infinite # of eigenvalues

$\tan(\lambda L)=\dfrac{Bi}{\lambda L}$ where $Bi=\dfrac{\bar{h}L}{k}$ we can identify eigenvalue range by plotting the residual residual $=\dfrac{Bi}{\lambda L}-\tan(\lambda L)$ Go to EES

Slowgraph Occurs if $(i-1)\pi < \lambda_i L < (i-1)\pi + \frac{\pi}{2}$

→ Setup range & guess values for each eigenvalue

```
Nterm = 10 [-]
duplicate i=1, Nterm
    lowerlimit[i] = (i-1)π
    upperlimit[i] = (i-1)π + π/2
    guess[i] = (i-1)π + π/4
end
```

→ implement eigencondition

```
duplicate i=1, Nterm
    tan(LambdaL[i]) = Bi/LambdaL[i]
    Lambda[i] = LambdaL[i]/L
end
```

→ variable info window

guess	lower	upper
guess[]	lowerlimit[]	upperlimit[]

Solve & Check Arraystable

⇒ Now solve the non homogeneous problem for each eigenvalue

$$\frac{d\theta_{ti}}{dt} + \lambda_i^2 \alpha \theta_{ti} = 0 \xrightarrow{\text{Spot the solution}} \theta_{ti} = C_{3i} \exp(-\lambda_i^2 \alpha t)$$

$$\Rightarrow \theta_i = \theta_{xi}\theta_{ti} = C_2 \cos(\lambda_i x)\, C_{3i} \exp(-\lambda_i^2 \alpha t) \Rightarrow$$

$$\Rightarrow \theta_i = C_i \cos(\lambda_i x) \exp(-\lambda_i^2 \alpha t)$$

→ Apply IC: $\theta_{t=0} = T_{ini} - T_\infty \Rightarrow \sum_{i=1}^{\infty} C_i \cos(\lambda_i x) = T_{ini} - T_\infty$

Use orthogonality of eigenfunctions

$$\sum_{i=1}^{\infty} C_i \underbrace{\cos(\lambda_i x)\cos(\lambda_j x)}_{=0 \text{ if } i \neq j} = (T_{ini} - T_\infty)\cos(\lambda_j x)$$

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$$C_i \int_0^L \cos^2(\lambda_i x)\,dx = (T_{ini} - T_\infty) \int_0^L \cos(\lambda_i x)\,dx \Rightarrow$$

$$\Rightarrow \frac{C_i}{2} \frac{\cos(\lambda_i L)\sin(\lambda_i L) + \lambda_i L}{\lambda_i} = (T_{ini} - T_\infty) \frac{\sin(\lambda_i L)}{\lambda_i}$$

$$C_i = \frac{2(T_{ini} - T_\infty)\sin(\lambda_i L)}{\cos(\lambda_i L)\sin(\lambda_i L) + \lambda_i L} \;\; \& \;\; \theta_i = C_i \cos(\lambda_i x)\exp(-\lambda_i^2 \alpha t)$$

⇒ show solution in EES plot

Fourier # $F_o = \frac{\alpha t}{L^2}$

→ @ $F_o < 0.2$ thermal wave has not hit wall & semi infinite solution could apply

→ @ $F_o > 0.2$ exact solution can be approximated by single term

→ show plot Pg. 417

→ @ $B_i \simeq 0.1$ exact solution corresponds to Lumped Capacitance solution