Lesson 10: 1-D Analytical Transients

Last time we discussed lumped capacitance transient problems. These assumed no temperature gradients within the material, 1-D Transients problems cannot be treated as lumped because the Biot number is not sufficiently small.

The simplest 1-D problem is the semi-infinite body — material is bounded on one edge and extends "infinitely" in the other direction. Nothing is truly semi-infinite, but the model is usually appropriate for short times.

We will use 2 approaches: 1) Approximate model helps us understand the problem.

2) Self-similar solution

At time $t_1$, by applying Fourier's Law, $kA \frac{dT}{ds}$.

Imagine heating a material instantly at $T_s$. A thermal wave will begin to penetrate into the material. The depth of the penetrating $S_t$ grows with time.

Approximate model: uses resistances to help us understand how energy is conducted into the material and how energy is stored in the material that has been heated.

Approximate conduction:

$$G_{cond} = \frac{(T_s - T_{ini})}{T_s - T_{ini}}$$

Approximate storage:

$$V = A \cdot \frac{1}{2} \cdot \frac{(T_s - T_{ini})}{T_s - T_{ini}}$$

Total heat capacity of material $= A \cdot \frac{1}{2} \cdot \frac{(T_s - T_{ini})}{T_s - T_{ini}}$.

Average heat rise of material in CV $= A \cdot \frac{1}{2} \cdot \frac{(T_s - T_{ini})}{T_s - T_{ini}}$.

Rate of conduction decreases as $S_t$ grows.

$$\frac{dT}{ds} = \frac{kA}{2} \cdot \frac{(T_s - T_{ini})}{T_s - T_{ini}}$$

$V$ grows because more material is being heated ($S_t$ is growing).
Energy Balance: $\frac{dV}{dt} \rightarrow kA \left( \frac{T_f - T_i}{s} \right) \frac{\Delta P}{C} \left( \frac{s - T_i}{2} \right) \frac{dS}{dt} \rightarrow$

$\rightarrow 2 \left( \frac{k}{C} \right) S \frac{dS}{dt} \rightarrow$ This group slows up often in transient conduction problems $\frac{1}{\tau} = \frac{k}{C}$ thermal diffusivity

$\rightarrow 2 \left( \frac{k}{C} \right) S \frac{dS}{dt} \rightarrow 2 \frac{dS}{dt} \frac{s}{2} \rightarrow s = 2l^2t$

$\rightarrow$ Transient conduction problems behave according to this. Conduction distance depends on time & thermal diffusivity, leads to

$S_t = 2\sqrt{\alpha t}$

$\Rightarrow L = 2\sqrt{\alpha S}$

$\Rightarrow \frac{\Delta t}{\Delta x} = \frac{L^2}{4 \alpha}$

$\Rightarrow$ Special case of object

Whenever you are confronted with a transient conduction problem after you calculate the Biot, calculate two time constants to understand problem:

1) Time for external equilibration: lumped capacitance time constant. This is approximately the time required for the object to equilibrate with its surroundings $\tau_{lumped} = \frac{1}{R C}$ (total heat capacity)

2) Time for internal equilibration: diffusive time constant. This is approximately the time required for the object to equilibrate internally by conduction $\tau_{diff} = \frac{L^2}{4 \alpha}$
The Self-Similar Solution

An exact analytical solution to the semi-infinite body problem can be obtained using a self-similar solution which recasts the problem into one involving only a single similarity parameter $C$. It is only possible if both the PDE and BCs can be cast into a form.

Example: Semi-infinite problem with $T_{0}$

Step 1: Define CE

Step 2: Energy Balance $\dot{q} = \dot{q}_x dx + \dot{q}_t$

Step 3: Take limit as $dx \to 0$

$\dot{q}_x = \dot{q}_t dx \to 0 = \frac{d\dot{q}_x}{dx} dx + \frac{d\dot{q}_t}{dt}$

Step 4: Rate Equations

$\dot{q}_x = -k \frac{dT}{dx} \frac{dT}{dx} = \rho c A \frac{dT}{dt}$

$0 = \frac{d}{dx} \left[ -k \frac{dT}{dx} \right] dx + \rho c \frac{A}{dx} \frac{dT}{dt} \to \oint dx \frac{d^2T}{dx^2} = \frac{dT}{dt} \frac{T_{x=0} - T_s}{T_{x=\infty} - T_{x=0}}$

Step 5: Select Similarity Parameter $S$ is clearly a function of $x$, $t$, and $T$

The temperature distribution is a function of any time is independent on $x$

This is the similarity variable for this problem

$T(x,t) = T(S(x,t))$

Step 6: Transform PDE $x \frac{d^2T}{dx^2} = \frac{dT}{dt}$, Gedeon's triangle, intensity of $T$

$\frac{d^2T}{dx^2} \to \frac{d}{dx} \left( \frac{dT}{dx} \right)$

$\frac{d}{dx} \left( \frac{dT}{dx} \right) \to \frac{d^2T}{dx^2} \frac{1}{2 \sqrt{T}}$

$x = \frac{1}{2 \sqrt{T}} \to \frac{d^2T}{dx^2} \frac{1}{2 \sqrt{T}} \to \frac{d^2T}{dx^2} \frac{1}{2 \sqrt{T}} \to \frac{d^2T}{dx^2} \frac{1}{2 \sqrt{T}}$

$\frac{d^2T}{dx^2} = \frac{d}{dx} \left( \frac{dT}{dx} \right) \to \frac{d}{dx} \left( \frac{dT}{dx} \right)$

$\frac{d}{dx} \left( \frac{dT}{dx} \right) \to \frac{d^2T}{dx^2} \frac{1}{2 \sqrt{T}}$
\[
\frac{dT}{t} - \frac{dN}{t} = \frac{x}{2\sqrt{\pi t}} \rightarrow \frac{dT}{dt} \rightarrow \frac{d^2T}{dx^2} = \frac{-x}{4\sqrt{\pi t} dN}
\]
Now substitute into PDE: \[\alpha \frac{d^2T}{dx^2} - \frac{dT}{t} \rightarrow \frac{d^2T}{dx^2} - \frac{1}{4\sqrt{\pi t} dN} \rightarrow \frac{d^2T}{dN^2} = -\frac{2\alpha}{dN} \]
Transform the BC's:
\[
T_0 = T_s \Rightarrow T_{x=0} = T_s
\]
\[
T_{x=0} = T_{ini} \Rightarrow T_{x=0} = T_{ini}
\]
\[
T_{x=\infty} = T_{ini} \Rightarrow T_{x=\infty} = T_{ini}
\]
Step 7: Solve the ODE
Define \[w = \frac{dT}{dN} \] and substitute:
\[
\frac{dw}{dN} = -\frac{2\alpha}{dN} w
\]
Separate:
\[
\frac{dw}{w} = -2\alpha dN \Rightarrow \int \frac{dw}{w} = -2\alpha \int dN \rightarrow \ln(w) = -2\alpha N + C
\]
Solve for \(w\): \[w = e^{C \cdot -2\alpha} \]
Rearrange: \[
\frac{dT}{dN} = \frac{C}{2} e^{C \cdot -2\alpha} \]
Separate again:
\[
\frac{dT}{C} = e^{C \cdot -2\alpha} dN \Rightarrow \int \frac{dT}{C} = \int e^{C \cdot -2\alpha} dN
\]
\[
T = T_s + C \int e^{C \cdot -2\alpha} dN
\]
This function is known as the Gaussian Error function,
\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt
\]
So:
\[
T = T_s + C \sqrt{\frac{2}{\pi}} \text{erf}(\alpha\sqrt{2/2})
\]
\[
C = \frac{2(T_{ini} - T_s)}{\sqrt{\pi}}
\]
\[
T = T_s + (T_{ini} - T_s) \text{erf}(\alpha \sqrt{2})
\]
\[
\alpha = \frac{kA}{\sqrt{\pi t_{01}}}
\]
Show alternatives in EES.