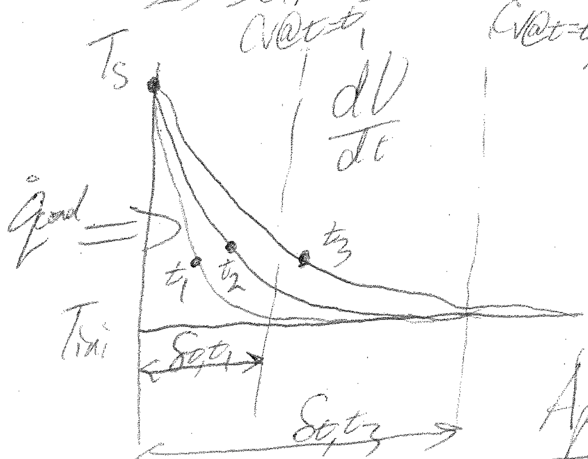
Lesson 10 1-D Analytical Transients

→ Semi-Infinite Solutions
  → Example
→ SOV
  → Example

→ Last time we discussed Lumped capacitance transient problems. These assumed no temperature gradients within the material. 1-D Transients problems cannot be treated as lumped because the Biot number is not sufficiently small.

→ The simplest 1-D problem is the semi-infinite body → material is bounded on one edge & extends "infinitely" in the other direction. Nothing is truly semi-infinite, but the model is usually appropriate for short times.

→ We will use 2 approaches 1) Approximate model → helps us understand the problem
2) Exact Self-similar solution



→ Imagine heating a material ~~[illegible]~~ instantly @ T_{ini} by applying heat to surface to T_S. A thermal wave will begin to penetrate into the material. The depth of the penetrating δ_t grows with time.

Approximate model: uses resistances to help us understand how energy is conducted into the material and how energy is stored in the material that has been heated.

Draw CV

Approximate conduction

$$\dot{q}_{cond} \approx \frac{(T_S - T_{ini})}{R_{semi-\infty}} \quad \text{where } R_{semi-\infty} = \frac{\delta_t}{kA_c} \Rightarrow \dot{q}_{cond} \approx kA_c \frac{(T_S - T_{ini})}{\delta_t}$$

rate of conduction decreases as δ_t grows

Approximate storage

$$U \approx \underbrace{A_c \delta_t \rho c}_{\text{Total heat capacity of material in CV}} \underbrace{\frac{(T_S - T_{ini})}{2}}_{\text{Average temp rise of material in CV}} \Rightarrow \frac{dU}{dt} \approx A_c \rho c \frac{(T_S - T_{ini})}{2} \frac{d\delta_t}{dt}$$

U grows because more material is being heated (δ_t is growing)

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Energy Balance: $\dot{q}_{cond} \approx \frac{dU}{dt} \Rightarrow kA_c \frac{(T_s - T_{ini})}{\delta_t} \approx A_c \rho c \frac{(T_s - T_{ini})}{2} \frac{d\delta_t}{dt} \Rightarrow$

$$\Rightarrow 2\left(\frac{k}{\rho c}\right) \approx \delta_t \frac{d\delta_t}{dt}$$

This group shows up often in transient conduction problems $\alpha\left(\frac{m^2}{s}\right) = \frac{k}{\rho c}$ thermal diffusivity

$$\Rightarrow 2\alpha \int_0^t dt \approx \int_0^{\delta_t} \delta_t \, d\delta_t \Rightarrow 2\alpha t \approx \frac{\delta_t^2}{2} \Rightarrow \delta_t \approx 2\sqrt{\alpha t}$$

→ Transient conduction problems behave according to this ↑ conduction distance depends on time & thermal diffusivity, & leads to

→ Diffusive time constant: time it takes for internal equilibration by conduction.

$$\underbrace{\delta_t}_{\text{spacial extent of object}} \approx 2\sqrt{\alpha \underbrace{t}_{\text{time required for conduction wave to move across object} = \tau_{diff}}} \Rightarrow L = 2\sqrt{\alpha \tau_{diff}} \Rightarrow \underbrace{\tau_{diff}}_{\text{diffusive time constant}} \approx \frac{L^2}{4\alpha}$$

☆ Whenever you are confronted with a transient conduction problem, after you calculate the Biot #, calculate two time constants to understand the problem:

1) time for external equilibration: lumped capacitance time constant → this is approximately the time required for the object to equilibrate with its surroundings

$$\tau_{lumped} = R_{sos}\, C \leftarrow \text{total heat capacity}$$

2) time for internal equilibration: diffusive time constant → this is approximately the time required for the object to equilibrate internally by conduction

$$\tau_{diff} = \frac{L^2}{4\alpha}$$

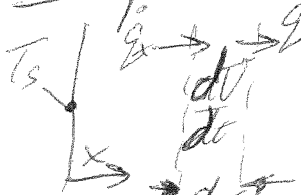
The Self-Similar Solution

→ An exact analytical solution to the semi-infinite body problem can be obtained using a self-similar solution

→ recasts problem involving 2 variables (e.g. x & t) into a problem involving only a single, similarity parameter (η)

→ only possible if both the PDE & BC's can be cast in terms of η

Example: Semi-infinite problem with T_{ini}



Step 1: Define CV

Step 2: Energy Balance $\dot q_x = \dot q_{x+dx} + \frac{\partial U}{\partial t}$

Step 3: Take limit as $dx \to 0$

$$\dot q_x = \dot q_x + \frac{\partial \dot q_x}{\partial x}dx + \frac{\partial U}{\partial t} \Rightarrow 0 = \frac{\partial \dot q_x}{\partial x}dx + \frac{\partial U}{\partial t}$$

Step 4: Rate Equations: $\dot q_x = -kA_c\frac{\partial T}{\partial x}$ $\quad \frac{\partial U}{\partial t} = \rho c A_c dx \frac{\partial T}{\partial t}$

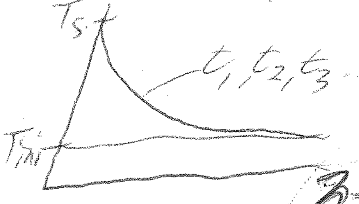
$$0 = \frac{\partial}{\partial x}\left[\frac{-k\cancel{A_c}\frac{\partial T}{\partial x}}{\rho c}\right]\cancel{dx} + \frac{\rho c \cancel{A_c}}{\rho c}\cancel{dx}\frac{\partial T}{\partial t} \Rightarrow \alpha\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

with $T_{x=0} = T_s$, $T_{x\to\infty} = T_{ini}$, $T_{t=0} = T_{ini}$

Step 5: Select Similarity Parameter → T is clearly a function of x & t but the temperature distribution @ any time is dependent on δ_t

this is the similarity variable for this problem

$$T(x,t) = T(\eta(x,t))$$



$$\eta = \frac{x}{\delta_t} = \frac{x}{2\sqrt{\alpha t}}$$

Step 6: Transform PDE: $\alpha\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$ & $\eta = \frac{x}{2\sqrt{\alpha t}}$ Get derivatives in terms of η

$$\frac{\partial T}{\partial x} = \frac{\partial[T(\eta(x,t))]}{\partial x} \Rightarrow \frac{dT}{d\eta}\boxed{\frac{\partial \eta}{\partial x}} \quad \frac{\partial \eta}{\partial x} = \frac{1}{2\sqrt{\alpha t}} \Rightarrow \frac{\partial T}{\partial x} = \frac{dT}{d\eta}\frac{1}{2\sqrt{\alpha t}}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{d}{d\eta}\left(\frac{\partial T}{\partial x}\right)\frac{\partial \eta}{\partial x} \Rightarrow \frac{\partial^2 T}{\partial x^2} = \frac{d}{d\eta}\left(\frac{dT}{d\eta}\frac{1}{2\sqrt{\alpha t}}\right)\frac{1}{2\sqrt{\alpha t}} \Rightarrow \frac{\partial^2 T}{\partial x^2} = \frac{d^2T}{d\eta^2}\frac{1}{4\alpha t}$$

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$\frac{\partial T}{\partial t} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial t} \rightarrow \eta = \frac{x}{2\sqrt{\alpha t}} \Rightarrow \frac{\partial \eta}{\partial t} = -\frac{x}{4t\sqrt{\alpha t}} \Rightarrow \frac{\partial T}{\partial t} = -\frac{x}{4t\sqrt{\alpha t}}\frac{dT}{d\eta}$

Now substitute into PDE: $\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \Rightarrow \alpha \frac{d^2T}{d\eta^2}\frac{1}{4\alpha t} = -\frac{x}{4t\sqrt{\alpha t}}\frac{dT}{d\eta} \Rightarrow$

$\Rightarrow \frac{d^2T}{d\eta^2} = -2\underbrace{\frac{1}{2\sqrt{\alpha t}}}_{\eta}\frac{dT}{d\eta} \Rightarrow \underline{\underline{\frac{d^2T}{d\eta^2} = -2\eta\frac{dT}{d\eta}}}$ We transformed the PDE for $T(x,t)$ into the ODE for $T(\eta)$.

Transform the BC's: $T_{x=0,t} = T_s \Rightarrow T_{\eta=0} = T_s$

$\eta = \frac{x}{2\sqrt{\alpha t}}$ $\quad T_{x,t=0} = T_{ini} \Rightarrow T_{\eta\to\infty} = T_{ini}$

$T_{x\to\infty,t} = T_{ini} \Rightarrow T_{\eta\to\infty} = T_{ini}$

$\frac{d^2T}{d\eta^2} = -2\eta\frac{dT}{d\eta}$

Step 7: Solve the ODE define $w = \frac{dT}{d\eta}$ & substitute: $\frac{dw}{d\eta} = -2\eta w$

Separate: $\frac{dw}{w} = -2\eta\, d\eta$ & integrate: $\int \frac{dw}{w} = -2\int \eta\, d\eta \Rightarrow \ln(w) = -\eta^2 + C_1$

Solve for w: $w = \exp(-\eta^2 + C_1)$, rearrange: $\frac{dT}{d\eta} = C_2 \exp(-\eta^2)$

Separate again: $dT = C_2 \exp(-\eta^2)\,d\eta$ & integrate again: $\int_{T_s}^{T} dT = C_2 \int_0^{\eta} \exp(-\eta^2)\,d\eta$

$\Rightarrow T = T_s + C_2 \underbrace{\int_0^{\eta} \exp(-\eta^2)\,d\eta}$ this function is known as the Gaussian Error function

& complimentary error function $\mathrm{erfc}(\eta) = 1 - \mathrm{erf}(\eta)$ $\quad \mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_0^{\eta} \exp(-\eta^2)\,d\eta$

So $T = T_s + C_2 \frac{\sqrt{\pi}}{2}\mathrm{erf}(\eta)$ apply $T_{\eta\to\infty} = T_{ini} \Rightarrow T_s + C_2\frac{\sqrt{\pi}}{2}\underbrace{\mathrm{erf}(\infty)}_{1} = T_{ini}$

so $C_2 = \frac{2(T_{ini} - T_s)}{\sqrt{\pi}}$ & $\underline{T = T_s + (T_{ini} - T_s)\,\mathrm{erf}(\eta)}$ & $\dot{q}_{x=0} = \frac{kA_c}{\sqrt{\pi\alpha t}}(T_s - T_{ini})$

Show alternatives in EES