0-D Transients

→ Until now we have been solving steady state problems (no T vs. time) such that there was no change in energy storage 1-D: $T(x)$; 2-D: $T(x,y)$

→ With transient problems temperature changes with time & energy storage such that 0-D: $T(t)$; 1-D: $T(x,t)$; 2-D: $T(x,y,t)$; etc.

☆ 0-D transients are known as "Lumped Capacitance" problems. All of the material is "lumped" together & placed at the same temperature. Similar to our finned/extended surface approximation, we use the Biot number to determine whether a Lumped Capacitance approximation is justified:

$$B_i = \frac{\text{Resistance you want to neglect}}{\text{Resistance you want to include}} \Rightarrow B_i = \frac{R_{\text{internal conduction}}}{R_{\text{surface-to-surroundings}}}$$

→ if $B_i \ll 1$ then a lumped capacitance approach is justifiable.

☆ Lumped capacitance problems usually involve finite objects that are not infinitely long. Estimating the resistances for conduction is not always easy. It is commonly accepted that: $R_{\text{internal conduction}} = \frac{L_{\text{conduction}}}{k A_s}$ where

$$L_{\text{conduction}} = \text{average length of conduction} = \frac{\forall \leftarrow \text{volume}}{A_s \leftarrow \text{surface area}}$$

$L_{\text{conduction}}$ then approximates the shortest conduction path.

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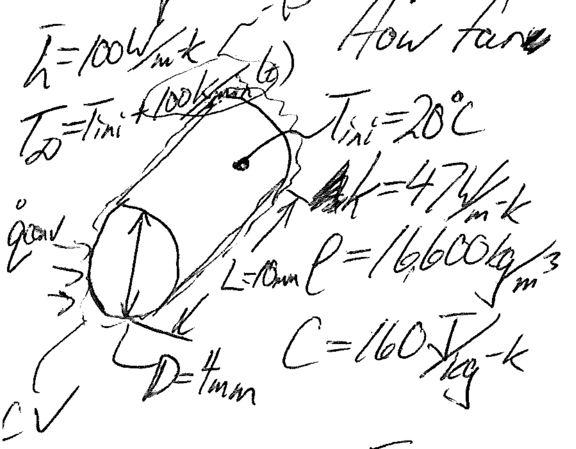
Example: Temperature sensor exposed to changing fluid temp.
How far behind does the sensor lag? $\Rightarrow$ What is the measurement error?



Step 0: Simple resistance calcs

$$B_i = \frac{R_{internal}}{R_{surface\,to\,surroundings}} \Rightarrow R_{internal} = \frac{L_{conduction}}{k A_s}$$

$$R_{stos} = \frac{1}{\bar{h} A_s}$$

$$L_{conduction} = \frac{\forall}{A_s} = \frac{\frac{\pi}{4} D^2 L}{\frac{\pi}{2} D^2 + \pi D L} \Rightarrow \underline{B_i = 0.002 \ll 1}$$ Lumped Capacitance justified

Step 1: Define system & assumptions: 1) lumped, 2) No generation, 3) uniform...

Step 2: Energy Balance: $\dot{E}_{in} = \dot{E}_{out} + \frac{dE_{stored}}{dt} \Rightarrow \dot{q}_{conv} = \boxed{\frac{dU}{dt}}$ rate change of internal energy with time.

Step 3: Rate Equations: $\dot{q}_{conv} = \bar{h} A_s (T_\infty - T) \Rightarrow$ Convection term

Storage term: $\frac{dU}{dt} = m \frac{du}{dt} \Rightarrow \frac{dU}{dt} = \rho \forall \frac{du}{dT} \frac{dT}{dt} \Rightarrow \frac{dU}{dt} = \rho \forall C \frac{dT}{dt}$

mass   specific internal energy   heat capacity

$$\frac{\bar{h} A_s (T_\infty - T) = \rho \forall C \frac{dT}{dt}}{\rho \forall C} \Rightarrow \underline{\frac{dT}{dt} + \frac{\bar{h} A_s}{\rho \forall C} T = \frac{\bar{h} A_s}{\rho \forall C} T_\infty}$$

where $T_\infty = T_{ini} + \beta t$

Governing ODE

Notice what's controlling temperature here with time: $\frac{\bar{h} A_s}{\rho \forall C}$

This group occurs so often with transient problems that it is called the Lumped Capacitance Time Constant: $\tau_{lumped} = \frac{mC}{\bar{h} A_s}$

$\tilde{\tau}_{lumped}$ is a very useful parameter for understanding heat transfer problems. $\tilde{\tau}_{lumped}$ is approximately the amount of time required for the object to respond to a change in surroundings.

for this problem: $\tilde{\tau}_{lumped} = \dfrac{\rho V c}{h A_s} \approx \underline{22s}$!!

& our ODE becomes: $\dfrac{dT}{dt} + \dfrac{T}{\tilde{\tau}_{lumped}} = \dfrac{(T_{ini} + \beta t)}{\tilde{\tau}_{lumped}}$ 1st order Linear non homogeneous ODE

<u>Step 4</u>: Solve the ODE (split into homogeneous & particular components)

$T = T_h + T_p$ so $\dfrac{dT_h}{dt} + \dfrac{T_h}{\tilde{\tau}_{lumped}} = 0$ & $\dfrac{dT_p}{dt} + \dfrac{T_p}{\tilde{\tau}_{lumped}} = \dfrac{(T_{ini} + \beta t)}{\tilde{\tau}_{lumped}}$

<u>Homogeneous Solution</u>

$T_h = C_1 \exp(at)$ → substitute into homogeneous ODE

$$\frac{d}{dt}\left[C_1 \exp(at)\right] + \frac{C_1 \exp(at)}{\tilde{\tau}_{lumped}} = 0 \Rightarrow \cancel{C_1} a \cancel{\exp(at)} + \frac{\cancel{C_1}}{\tilde{\tau}_{lumped}} \cancel{\exp(at)} = 0$$

So $a = \dfrac{1}{-\tilde{\tau}_{lumped}}$ & $T_h = C_1 \exp\left(\dfrac{-t}{\tilde{\tau}_{lumped}}\right)$

<u>Particular Solution</u>: $\dfrac{dT_p}{dt} + \dfrac{T_p}{\tilde{\tau}_{lumped}} = \dfrac{1}{\tilde{\tau}_{lumped}}(T_{ini} + \beta t)$

By inspection we know that the solution must be a linear function: $T_p = C_2 + C_3 t$

$$\frac{d}{dt}\left[C_2 + C_3 t\right] + \frac{C_2 + C_3 t}{\tilde{\tau}_{lumped}} = \frac{T_{ini} + \beta t}{\tilde{\tau}_{lumped}} \Rightarrow C_3 + \frac{C_2 + C_3 t}{\tilde{\tau}_{lumped}} = \frac{T_{ini} + \beta t}{\tilde{\tau}_{lumped}}$$

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Use the **Method of Undetermined Coefficients**: Divide into constant & linear terms

Constant terms $C_3 + \frac{C_2}{\tilde{\tau}_{lumped}} = \frac{T_{ini}}{\tilde{\tau}_{lumped}}$ linear terms $\frac{C_3 t}{\tilde{\tau}_{lumped}} = \frac{\beta t}{\tilde{\tau}_{lumped}} \Rightarrow C_3 = \beta$

$$\beta + \frac{C_2}{\tilde{\tau}_{lumped}} = \frac{T_{ini}}{\tilde{\tau}_{lumped}} \Rightarrow C_2 = T_{ini} - \beta\tilde{\tau}_{lumped} \text{ so } T_p = T_{ini} - \beta\tilde{\tau}_{lumped} + \beta t$$

$T = T_h + T_p$ so $T = C_1 \exp\left(\frac{-t}{\tilde{\tau}_{lumped}}\right) + T_{ini} - \beta\tilde{\tau}_{lumped} + \beta t$ General Solution

Step 5: Apply Boundary Conditions; $T_{t=0} = T_{ini} = C_1 \exp\left(\frac{-0}{\tilde{\tau}_{lumped}}\right) + T_{ini} - \beta\tilde{\tau}_{lumped} + \beta 0 \Rightarrow$

$\Rightarrow T_{ini} = C_1 + T_{ini} - \beta\tilde{\tau}_{lumped} \Rightarrow C_1 = \beta\tilde{\tau}_{lumped}$ So

$$\boxed{T = \beta\tilde{\tau}_{lumped}\left[\exp\left(\frac{-t}{\tilde{\tau}_{lumped}}\right) - 1\right] + T_{ini} + \beta t}$$

Step 6: Now use software to implement & graph solution
Show in EES

Table 3-1 pg 306 in text has other handy solutions including Oscillating ambient temp & step change in temp.