Numerical Solution to 1-D Steady Conduction

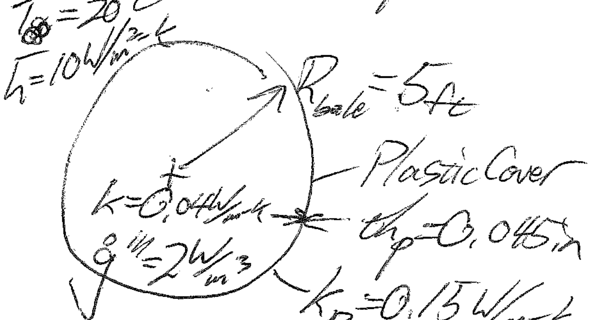
→ Limits to analytical
→ Integrated average k
→ Numerical solution
→ Define nodal network
→ Energy balance(s)
→ rate equations
→ numerical convergence
→ post analysis

→ Last time we introduced analytical solutions to Steady-State 1-D conduction problems.

→ Analytical solutions are very nice ✱ they are exact. ✱ they are computationally fast

→ Problem is, Analytical Solutions are not flexible.

__Back to the Haybale Example:__ When we solved this we assumed that properties in the hay bale are uniform, both throughout the bale & @ any temp. The thermal conductivity & volumetric generation are likely not constant & probably functions of both temperature & position due to moisture content & density



Our ODE from last time:

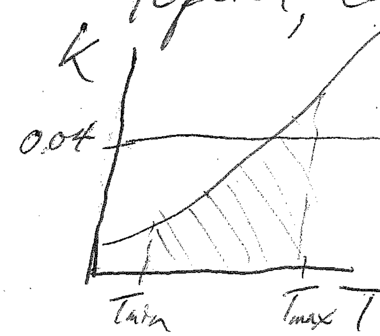
$$r\dot{g}''' = \frac{-d}{dr}\left(kr\frac{dT}{dr}\right)$$

~~.3048 = 1.1235~~ .075

What if $\dot{g}''' = 0.01T - 1$ & $k = 0.000015T^{1.3841}$?

With a just $\dot{g}'''$ we can still get analytical solution, but it involves Bessel functions. With k function it's just way too complicated.

→ 1 option, calculate the integrated average values



$$k_{int,ave} = \frac{\int_{T_{min}}^{T_{max}} k(T)\,dT}{(T_{max} - T_{min})}$$

Use EES integral command (numerically integrates)

$(T_{high} - T_{low})\, k_{int,ave} = \text{INTEGRAL}(\text{Integrand}, \text{Variable name}, \text{Lower limit}, \text{Upper limit}, \text{Step size})$

$0.000015T^{1.3841}$ T   293.2   322.3   Optional

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Plug into EES $k_{ave} = 0.04171 \frac{W}{m\text{-}K}$ $k_{high} = 0.0444$ $k_{low} = 0.039$

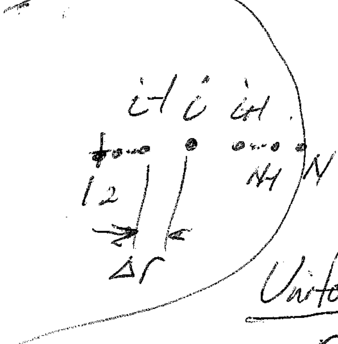
our analytical solution

This might get closer, but the spatial effects on T are not represented by the varrying k, & we'll have to iterate the solution for T_{bounds}. Really we need a different techique.

Numerical Solutions: Provide predictions of the temperature @ discrete locations (nodes) within the computational domain.

→ Numerical Solutions are approximate → Computationally slower

→ Flexible - easy to include temperature dependent properties



Step 1: Define many (N) nodes (locations where you will predict temperature) & the associated control volumes.

→ Small but finite (Δx is finite vs. dx approaches zero)

Uniformly distributed Nodes

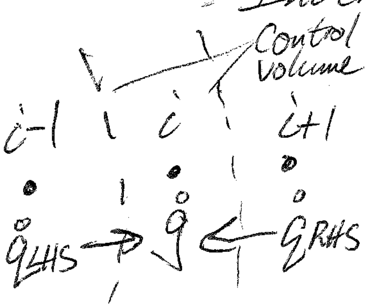
$$r_i = \frac{(i-1)}{(N-1)} R_{Bale} \quad \text{for } i = 1 \rightarrow N \qquad \Delta r = \frac{R_{Bale}}{(N-1)}$$

→ You can also write functions to concentrate nodes in regions of interest (e.g., regions with large temperature gradients

Step 2: Carry out an energy balance on each control volume

→ Internal nodes will look similar → boundaries must be treated seperately

Internal nodes:



IN = OUT + STORED → this is exactly correct energy must always balance

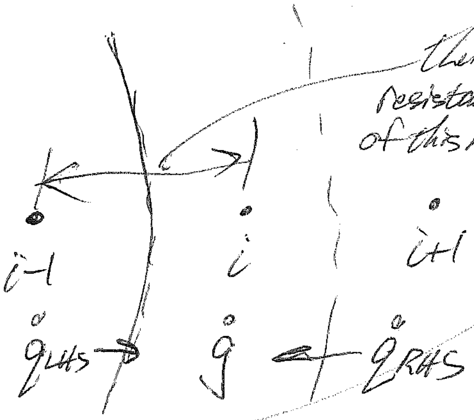
$$\dot{q}_{LHS} + \dot{q}_{RHS} + \dot{g} = 0$$

→ It doesn't matter whether heat transfers in or out of CV as long as I am consistent.

Step 3: Approximate each energy term with a rate equation

→ this step makes the numerical approximations possible

Internal nodes



$$\dot{q}_{LHS} + \dot{q}_{RHS} + \dot{q} = 0$$

$$\dot{q}_{LHS} \approx \frac{(T_{i-1} - T_i)}{R_{LHS}}$$

$(T_{i-1} - T_i)$ — heat transfer into the control volume

R_{LHS} — resistance of material separating nodes $i-1$ and i

$$R_{LHS} = \frac{\ln\left(\frac{r_i}{r_{i-1}}\right)}{2\pi k L}$$ → this will have issues @ $r_1 = 0$

Alternative approximation: $R_{LHS} = \dfrac{\text{conduction length}}{\boxed{k(\text{conduction area})}} = \dfrac{\Delta r}{k 2\pi L\left(r_i - \frac{\Delta r}{2}\right)}$

→ must be evaluated @ the interface

$$\dot{q}_{LHS} = \frac{2\pi L k}{\Delta r}\left(r_i - \frac{\Delta r}{2}\right)(T_{i-1} - T_i)$$

$$\dot{q}_{RHS} = \frac{2\pi L k}{\Delta r}\left(r_i + \frac{\Delta r}{2}\right)(T_{i+1} - T_i)$$

$$\dot{q}_{LHS} + \dot{q}_{RHS} + \dot{q} = 0$$

$$\dot{q} = \dot{q}''' 2\pi r_i L \Delta r$$

→ For all internal nodes this is $N-2$ equations in N unknowns $(T_1, \ldots T_N)$

→ The remaining 2 eqs. must come from the boundary nodes

Boundary Node 1:

Energy Balance $\dot{q}_{RHS} + \dot{q} = 0$ Rate Equations:

$$\dot{q}_{RHS} = \underbrace{2\pi L\left(\frac{\Delta r}{2}\right)}_{\text{Surface area}} \underbrace{\frac{k}{\Delta r}}_{\text{conduction length}} \underbrace{(T_2 - T_1)}_{\Delta T}$$

$$\dot{q} = \underbrace{\pi L\left(\frac{\Delta r}{2}\right)^2}_{\text{Volume of CV}} \dot{q}'''$$

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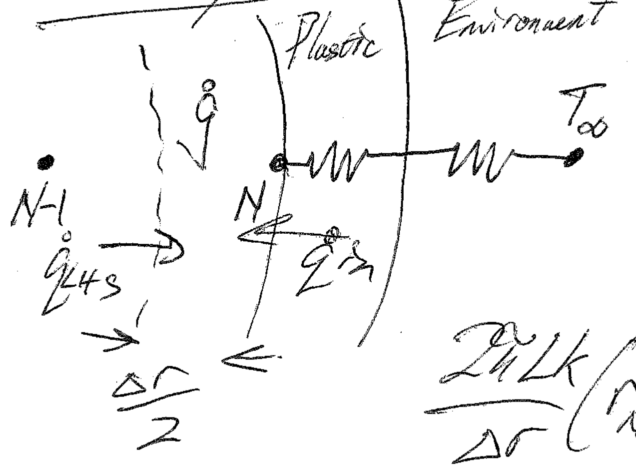
Boundary Node N:



Energy Balance: $\dot q_{LHS} + \dot q_{in} + \dot g = 0$

Rate Equations: $\dot q_{LHS} = 2\pi L\left(r_N - \frac{\Delta r}{2}\right)\frac{k}{\Delta r}\left(T_{N-1} - T_N\right)$

$$\dot q_{in} = \frac{(T_\infty - T_N)}{R_{cond,p} + R_{conv}} \qquad \dot g = 2\pi L r_N \frac{\Delta r}{2}\dot g'''$$

$$\frac{2\pi L k}{\Delta r}\left(r_N - \frac{\Delta r}{2}\right)\left(T_{N-1} - T_N\right) + \frac{(T_\infty - T_N)}{R_{cond,p} + R_{conv}} + \pi L r_N \Delta r \dot g''' = 0$$

Now we have a system of many algebraic equations (rather than a differential equation). It's easy to enter & solve these in EES.

Step 4: Implement model

Step 5: Verify that your model has numerically converged

→ Do you have enough Nodes? → Examine an import solution metric as N changes (parametric table)

→ Solution should stop changing as N becomes significantly large

→ Decision requires engineering judgement.

Step 6: Verify your solution makes sense.

→ show

Now for varying k & g, look@ where they are used. g is @ Node i. k is between nodes @ interface

$$\dot g_i''' = 0.01\,T_i - 1 \qquad k_{LHS,i} = 0.000015\left(\frac{T_{i-1}+T_i}{2}\right)^{1.3841} \qquad k_{RHS,i} = 0.000015\left(\frac{T_{i+1}+T_i}{2}\right)^{1.3841}$$