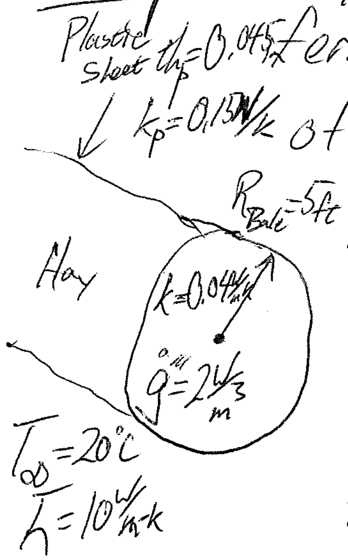
1-D Analytical Conduction

ME516F19 2.1

→ Last time we introduced the "lumped" property called thermal conductivity & showed that simple resistances can be incredibly useful for problem solving. However, resistance networks can ~~not~~ often not allow the level of detail & precision we need or desire. For simple problems we can develop analytical solutions that allow us to solve for the temperature @ any spatial location.

Example: A Hay Bale row has thermal energy generation due to bacterial fermentation (e.g. chemical reactions) where the magnitude of generation depends on the moisture content. If the hay bale is too large or wet, spontaneous combustion can occur when the temperature exceeds ~170°C. What is the centerline temp & how far from the end is this temp likely valid?



→ Go ahead & start to setup a procedure for solving this problem.

→ For those of you that had me for thermo, you'll remember my problem solving process steps that we used every time!

Step 1	Step 2	Step 3	Step 4	Step 5
Define the System being studied	1) List 2) Assumptions 3) About the 4) System that 5) Simplify balances	Mass, Entropy, Energy — Simplify Balances	Insert material/fluid property info	Solve, Plot, Report Out

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→ In Heat Transfer we need to modify this process some. In thermo the shape of our system or what was inside didn't really matter & we would purposefully draw our system boundaries to simplify the problem as much as possible. → as the point of HT is to know how properties like temperature vary within the system.

Step 0: Figure out what problem needs to be solved (0D, 1D, 2D, 3D) by doing simple resistances, dimensionless parameters, & time constant calcs.

$$R_{axial} = \frac{L}{kA_c} \Rightarrow \frac{1.524\,m}{0.04\frac{W}{m\text{-}k}\,\pi(1.524)^2} \Rightarrow \underline{5.22\frac{k}{W}} \qquad R_{conv} = \frac{1}{hA_s} \Rightarrow \frac{1}{10\frac{W}{m\text{-}k}\,2\pi 1.524_m} \Rightarrow \underline{0.01\frac{k}{W}}$$

$$R_{radial} = \frac{\ln\left(\frac{r_{out}}{r_{in}}\right)}{2\pi L k} \Rightarrow \frac{\ln\left(\frac{1.524\,m}{\approx 1\,m}\right)^{0.421}}{2\pi\, 1.524_m\, 0.04\frac{W}{m\text{-}k}}_{.383} \Rightarrow \underline{1.099} \qquad R_{plastic} = \frac{t_p}{kA_s} = \frac{0.00143_m}{0.15\frac{W}{m\text{-}k}\,2\pi 1.524_m}\ \underline{0.0079\frac{k}{W}}$$

R_{axial} & R_{radial} will see most ΔT. To determine @ what depth the end effect matters we can compare these by defining a new non-dimensional parameter

$$\Theta = \text{Hay Bale Parameter} = \frac{R_{axial}}{R_{radial}} \sim 1 \text{ then end effect is significant}$$

☆ All dimensionless parameters are just comparisons of two things of similar units.

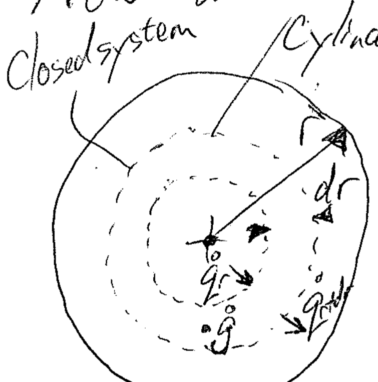
$$\Theta = \frac{\frac{L}{kA_c}}{\frac{\ln\left(\frac{r_{out}}{r_{in}}\right)}{2\pi L k}} \Rightarrow \frac{L}{\cancel{k}\pi r^2}\,\frac{2\pi L\cancel{k}}{\ln\left(\frac{r_{out}}{r_{in}}\right)} \Rightarrow 1 = \frac{2L^2}{r^2\ln\left(\frac{r}{1}\right)} \Rightarrow L = 0.7_m \approx 2.29\,ft$$

So end would matter if it was a single Bale, but we have a row.

As long as the row is >5 ft, a 1D radial solution to this problem should be fine.

→ Another change from thermo is that we now need to consider the generation of thermal energy → it's $\dot{q}$ just energy converted or degraded via entropy generation: → electrical into thermal via ohmic dissipation → nuclear into thermal ... etc.

Now we can move on to the next steps



Step 1: Define a system → differential control volume

Step 2: List Assumptions

1) Steady State (no time dependence)
2) Cylindrical, 1-D radial heat transfer (Temp gradient)
3) Internal heat generation & uniform (known) properties
4) Uniform boundary conditions

Step 3: Energy Balance: $E_{in} = E_{out} + \cancel{\Delta E_{stored}}^{\,0\,(1)} = \dot{q}_r + \dot{q} = \dot{q}_{r+dr}$

Simplifying our energy balance requires several substeps

SS 3.1: Expand term @ r+dr & take the limit as dr → 0

$$\cancel{\dot{q}_r} + \dot{q} = \cancel{\dot{q}_r} + \frac{d\dot{q}}{dr}dr \Rightarrow \dot{q} = \frac{d\dot{q}}{dr}dr$$

SS3.2: Substitute in the rate equations: Generation: $\dot{q} = \underbrace{d\forall}_{2\pi r L dr} \dot{q}'''$

Conduction: $\dot{q} = -2\pi r L k \frac{dT}{dr}$

$$\cancel{2\pi} r \cancel{L} dr \dot{q}''' = \frac{d}{dr}\left[-\cancel{2\pi} r \cancel{L} k \frac{dT}{dr}\right] dr \Rightarrow r\dot{q}''' = -k\frac{d}{dr}\left[r\frac{dT}{dr}\right]$$

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We now have a differential equation for temperature (not energy)

SS 3.3: Identify the type of differential equation

$\boxed{\text{Linear}}$ or non-Linear

1st Order
$\boxed{\text{2nd order}}$
3rd order
⋮

Homogeneous
$\boxed{\text{Non-homogeneous}}$

$\boxed{\text{Ordinary Diff Eq}}$
Partial Diff Eq.

SS 3.4: Solve the differential equation (Analytically or Numerically)

(This is where you should use software)

Seperate: $d\left[r\frac{dT}{dr}\right] = -\frac{r\dot{q}'''}{k}dr$

Integrate: $\int d\left[r\frac{dT}{dr}\right] = -\int \frac{r\dot{q}'''}{k}dr \Rightarrow r\frac{dT}{dr} = \frac{-r^2\dot{q}'''}{2k} + C_1$

Seperate Again: $\frac{dT}{dr} = \frac{-r\dot{q}'''}{2k} + \frac{C_1}{r} \Rightarrow dT = \left(\frac{-r\dot{q}'''}{2k} + \frac{C_1}{r}\right)dr$

Integrate Again: $\int dT = \int\left(-\frac{r\dot{q}'''}{2k} + \frac{C_1}{r}\right)dr \Rightarrow \underbrace{T = -\frac{r^2\dot{q}'''}{4k} + C_1\ln(r) + C_2}_{\text{General Solution to problem}}$

SS3.5: Apply Boundary Conditions @ $r=0$ & $r=R_{Bale}$

(We need this in temperature) $\dot{Q}_{r=0} = 0 \rightarrow \left(-2\pi r L k\frac{dT}{dr}\right)_{r=0} = 0$ — $r=0$ so satisfied if $\frac{dT}{dr}$ is bounded

Take derivative of general solution: $\frac{dT}{dr} = \frac{-r\dot{q}'''}{2k} + \frac{C_1}{r}$

Take limit as $r \to 0$ $\left(\frac{dT}{dr}\right)_{r=0} = \frac{-0\dot{g}'''}{2k} + \frac{C_1}{0}$ → this is bounded (not ∞) only if $C_1 = 0$.

So $T = \frac{-r^2 \dot{g}'''}{4k} + C_2$ Now apply 2nd BC @ $r = R_{Bale}$

where $\dot{q}_{r=R_{Bale}} = \dot{q}_{out}$

Now apply rate equations: $\dot{q}_{r=R_{Bale}} = -2\pi R_{Bale} L k \left(\frac{dT}{dr}\right)_{r=R_{Bale}}$

$T_{R_{Bale}} \circ\!-\!\overset{R_{plastic}}{\text{\/\/\/}}\!-\!\overset{R_{conv}}{\text{\/\/\/}}\!-\!\circ\, T_\infty$ $\xrightarrow{\dot{Q}_{out}}$

$\dot{q}_{out} = \frac{T_{R_{Bale}} - T_\infty}{R_{plastic} + R_{conv}}$ → $-2\pi R_{Bale} L k \left(\frac{dT}{dr}\right)_{r=R_{Bale}} = \frac{T_{R_{Bale}} - T_\infty}{\frac{th_p}{2\pi R_{Bale} L k_p} + \frac{1}{2\pi (R_{Bale} + th_p) L h}}$

Substitue into general solution

Take derivative $\left(\frac{dT}{dr}\right)_{r=R_{Bale}} = \frac{-R_{Bale}\dot{g}'''}{2k}$ ⟹ $-2\pi R_{Bale} L \frac{-R_{Bale}\dot{g}'''}{2k} = \frac{T_{R_{Bale}} - T_\infty}{\frac{th_p}{2\pi R_{Bale} L k_p} + \frac{1}{2\pi (R_{Bale} + th_p) L h}}$

Solve this for $T_{R_{Bale}}$ then solve for C_2 in general solution $C_2 = 322.3k$

So $T = \frac{-r^2 \dot{g}'''}{4k} + 322.3k$ @ centerline $r = 0$ so $T = 322.3k$

We can now plot this using software like EES.