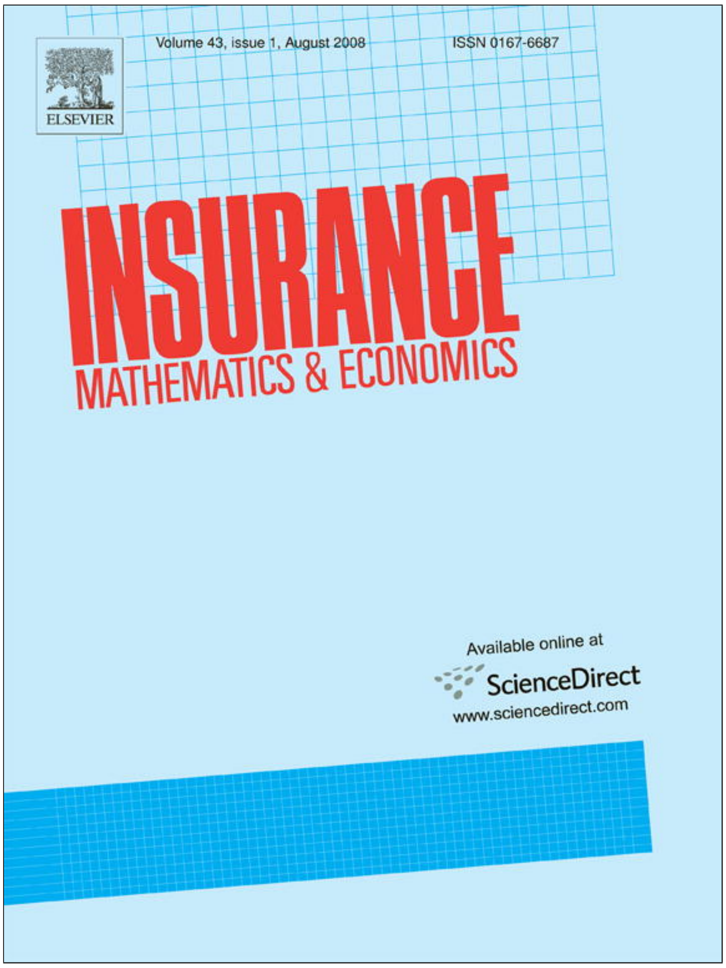
Volume 43, issue 1, August 2008
ISSN 0167-6687
ELSEVIER
INSURANCE
MATHEMATICS & ECONOMICS
Available online at
ScienceDirect
www.sciencedirect.com

Insurance: Mathematics and Economics 43 (2008) 108–115

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Threshold control of mutual insurance with limited commitment

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ARTICLE INFO

Article history:
Received June 2007
Received in revised form
October 2007
Accepted 26 March 2008

JEL classification:
C61
G22

Keywords:
Stochastic programming
Mutual insurance
Threshold control
Limited commitment
Diffusion process

ABSTRACT

We study the optimal premium policy of mutual insurance when the charged premium cannot be higher than a preset rate. We provide a complete solution to the problem and use numerical simulations to illustrate how the optimal premium policy responds to changes of outside factors. The results are useful for mutual insurance firms to design premium policies and can be used to test the behavior of these firms in empirical studies.

1. Introduction

This paper provides a complete solution to the stochastic control problems of mutual insurance and uses numerical simulations to illustrate how the solution responds to changes of outside factors. As such the paper is an extension of early works on mutual insurance done by Tapeiro (1984) and Tapiero and Jacque (1987, 1990), and also an extension of related works on stochastic control of general insurance done by Asmussen and Taksar (1997), Højgaard and Taksar (1998, 2001), Taksar and Markussen (2003), Cadenillas et al. (2006), and Taksar and Hunderup (2007).

Mutual insurance firms, in which individuals pool their risks to reduce insurance costs, coexist with the investor-owned or stock insurance firms and constitute a large part of the insurance industry.[1] As summarized by Smith and Stutzer (1990), the major differences between mutual and stock insurance firms are: (i) mutual insurance firms are owned by their policyholders, while stock insurance firms are owned by outside investors; (ii) mutual insurance firms have no capital, while stock insurance firms, similar to normal for-profit firms, need capital to operate; (iii) policyholders of stock insurance firms are charged a fixed premium, while the net price of a mutual policy is not known until *ex post* dividend payments are made; (iv) managers of mutual insurance firms are paid by expense savings, while managers of stock insurance firms are paid by investment profits. Such organizational differences lead to the general belief that mutual insurance firms are less efficient than stock insurance firms because in mutual insurance firms: (i) managers are commonly self-appointing and free of supervision from outside investors; (ii) managers have no incentive to maximize capital gain or profits. One question is then why mutual insurance can coexist with stock insurance; and answers to this question mainly include:

- Agency problems. It is argued by Mayers and Smith (1981) and Fama and Jensen (1983a,b) that mutual insurance firms have advantages over stock insurance firms in avoiding conflicts of interests between policyholders and investors, because policyholders and owners are the same.
- Idiosyncratic risks. Cass et al. (1996) show that mutual insurance can achieve economic efficiency when risks are idiosyncratic or individual.
- Sorting out different risks. Smith and Stutzer (1990) show that the coexistence of mutual and stock insurance firms serves as a sorting mechanism when individuals' risks are different; individuals with high probability to file claims will choose stock insurance and individuals with low probability to file claims will choose mutual insurance.

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[1] Mutuals account for 31% of life insurer assets, 21% of property-liability insurer assets, and 31% of savings and loan association assets (Remmers, 2003).

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doi:10.1016/j.insmatheco.2008.03.011

- Sorting out different preferences toward risk. In this case, the coexistence of mutual and stock insurance firms serves as a sorting mechanism to sort out heterogeneous customers; customers with high risk aversion will choose mutual insurance and customers with low risk aversion will choose stock insurance, as shown by Remmers (2003).

Our observations on marine insurance clubs confirm largely the above explanations. Marine insurance clubs are typical mutual insurance firms which cover liability costs of accidents of ship owners.[2] Accidents are idiosyncratic and are not at the market level; accidents do not happen frequently, but ship owners can be hurt severely by liability costs once an accident happens; members of marine insurance clubs are normally homogeneous in terms of the fleet size, and the homogeneity of members has the advantage of spreading risks and reducing agency costs caused by conflicts of interests among members (Hansmann, 1985).

As mentioned, unlike the fixed premium in stock insurance, the net price of a mutual policy is unknown until the *ex post* dividends are made. In practice, mutual insurance firms generally set a nominal premium rate which is set high enough (as a *priori*) to provide sufficient reserves for the pessimistic forecasts of claims; the excessive reserves can be returned fully or partially to policyholders as dividends; on the other hand, if the reserves provided by the nominal premium rate are inadequate because of worse-than-forecasted events, renegotiations among policyholders are required and mutual insurance firms are likely to dissolve because the participation of mutual insurance is voluntary.[3] The nominal premium rate can then be considered as the limited commitment of policyholders to mutual insurance firms; charging more than the nominal rate can be very costly to the management. Full risk pooling mutual insurance, in which policyholders share the costs whenever claims come, can hardly exist in reality without legal enforcement.

Motivated by these observations, in this paper we investigate the optimal control of mutual insurance with imperfect risk sharing. We first formulate the problem faced by the managers of a representative mutual insurance firm as a continuous-time stochastic dynamic programming model and then provide a complete solution to the problem; the comparative statics of the solution are studied via numerical simulations because the solution is expressed in terms of special functions which cannot be evaluated analytically.

2. The model

We consider a mutual insurance firm with finite individual policyholders $I = [1, \ldots, n]$, and each of the policyholders faces a risk over time. The set of the policyholders' risks at time period t is denoted by $X_t = [x_{1t}, \ldots, x_{nt}]$. The dynamics of each claim process are modeled as a stochastic diffusion process based on Brownian motion with drift

$$\mathrm{d}x_{it} = \gamma \mathrm{d}t + \delta \mathrm{d}w_{it} \quad i \in I \tag{1}$$

where γ is the arrival rate, and δ^2 is the variance rate of the claim; $\mathrm{d}w_{it}$ represents a differential of an adapted Wiener process. Under this specification, over a time interval Δt, the change in x_i, denoted by Δx_i, is characterized by a normal distribution with the mean of $\gamma \Delta t$ and the variance of $\delta^2 \Delta t$. Since one of the main reasons for individuals to choose mutual insurance is that their risks are idiosyncratic rather than collective, the elements in X_t are modeled independent of each other. Therefore the dynamics of the aggregate risk of the members are

$$\mathrm{d}x_t = \lambda \mathrm{d}t + \sigma \mathrm{d}w_t \tag{2}$$

with $\lambda = n\gamma$, and $\sigma^2 = n\delta^2$.

The actual premium rate of the mutual insurance policy is denoted by q_t, and the mutual insurance firm's net reserves (y_t) in a time interval $[t, t + \Delta t]$ evolve as

$$y_{t+\Delta t} = y_t + (nq_t - c - \rho y_t)\,\Delta t - \Delta x_t. \tag{3}$$

Let $\Delta t \rightarrow 0$, the process can be expressed by the stochastic differential equation

$$\mathrm{d}y_t = (nq_t - c - \lambda - \rho y_t)\,\mathrm{d}t + \sigma \mathrm{d}w_t \tag{4}$$

where nq_t is the aggregate premium rate; $c + \rho y_t$ denotes the operation cost rate in which c is the fixed component and ρy_t is the variable component with ρ denoting the positive unit cost of carrying positive net reserves (excessive reserves).[4] Excessive reserves can be invested and in such a case ρ can be negative to represent the unit investment profit. Although the sign of ρ itself is not important in our analysis, incorporating investment decisions into the model will be problematic. First of all, managers of mutual insurance firms are mainly paid by expense savings rather than investment profits, and their objectives are better characterized by minimizing the aggregate insurance costs of policyholders. Second, theories suggest that individuals choose mutual insurance because of their risk-averse preferences. Investment profits can hardly be one of the main objectives of risk-averse individuals.

The actual premium rate cannot be larger than the nominal one denoted by q^+. Given the nominal premium rate, nq^+ represents the premium commitment of policyholders to the mutual insurance firm. We use q^- to denote the maximal dividend rate (or the minimal premium rate); a smaller value of q^- represents a larger dividend rate and a negative value of q^- means paying positive dividends. For example, $q^- = 0$ means that the mutual insurance firm never pays dividends; $q^- = -\frac{y_t}{n}$ means that all excessive reserves are returned to policyholders as dividends.[5]

Starting from time s with a given initial reserve $y_s \equiv y_0 > 0$, the mutual insurance firm is said to be ruined at the first instant of time (τ) when its net reserves vanish, that is,

$$\tau = \inf\{t : y_t \le 0; t > s\}. \tag{5}$$

By this definition, at the ruin time the sum of excessive reserves and premium revenue is not enough to cover claims and policyholders refuse to pay any amount more than their maximal commitment. Each policyholder will thus have to choose the next best one to the mutual insurance to protect his risk. Let κ_t denote the premium rate that the policyholders would pay for the next best insurance, the resulted aggregate insurance costs are then $K = \int_s^{+\infty} n\kappa_t \mathrm{e}^{-r(t-s)}\mathrm{d}t$ with r denoting the discount rate. K can be basically interpreted as the bankruptcy costs of the mutual insurance, because the policyholders will pay that amount for their insurances if the mutual insurance firm is bankrupt.

The aggregate insurance cost minimization problem faced by a representative mutual insurance firm can be finally stated as

$$V(y_s) \equiv \mathrm{Min}_{q_t \in [q^-, q^+]} E\left(\int_s^{\tau} nq_t \mathrm{e}^{-r(t-s)}\mathrm{d}t + K\mathrm{e}^{-r(\tau-s)}\right) \tag{6}$$

$$s.t.\mathrm{d}y_t = (nq_t - c - \lambda - \rho y_t)\,\mathrm{d}t + \sigma \mathrm{d}w_t$$

$$y_t \ge 0, \quad \forall t \in [s, \tau)$$

y_s is given.

[2] The property costs of accidents of ship owners are generally covered by stock insurance.

[3] Such cases can be observed in the marine mutual insurance industry.

[4] The unit cost ρ captures the opportunity and administration costs of carrying positive reserves.

[5] The maximal dividend rate varies with time in this case because excessive reserves vary with time. The aggregated maximal dividend rate (nq^-) cannot be more than the excessive reserves.

The objective function in Eq. (6) is just the expected present value of the aggregate insurance costs, and it includes two components: the life-time paid premium $\int_s^{\tau} nq_t e^{-r(t-s)}dt$ and the bankruptcy costs incurred at the ruin time $K \equiv \int_{\tau}^{+\infty} n\kappa_t e^{-r(t-\tau)}dt$. The future costs are discounted by a discount rate r. The optimized value from the cost minimization problem is the mutual insurance firm's expected insurance cost function given an initial reserve y_s, which is denoted by $V(y_s)$. Since the ruin time is random, the cost minimization problem here is an example of the random-stopping problem discussed in Boukas et al. (1990).

It is worthwhile to point out that both the nominal premium rate (q^+) and the dividend policy (q^-) should be decision variables in a more complete model incorporating policyholders' both insurance and production decisions. Policyholders can reduce insurance costs by increasing their commitments,[6] but doing so will hurt their productive activities because fewer resources will be allocated to such activities. The nominal premium rate can be set lower when paying fewer dividends. For example, consider the following two cases: (i) the mutual insurance firm returns all excessive reserves to its policyholders; (ii) the mutual insurance firm never pays dividends ($q^- = 0$). The nominal premium rate can be set lower in the second case because all excessive reserves from one period will be carried to the next period and thus the mutual insurance firm relies less on collecting premium to cover claims. However, the insurance cost will be higher in the second case because the mutual insurance firm has to pay costs to carry the excessive reserves. Conceptually, the optimal commitment and dividend policy can be found when marginal benefits (reduction of insurance costs) equal marginal costs (reduction of profits from productive activities), and the solutions are affected by outside factors such as bankruptcy costs.[7]

Solution

By standard results of Dynamic Programming, under regular conditions, the expected insurance cost function $V(y_s)$ exists and is the unique solution from the following Hamilton–Jacobi–Bellman (HJB) equation, which is a second-order ordinary differential equation (ODE) defined below

$$\frac{1}{2}\sigma^2 V''(y_s) + H\left(y_s, V'(y_s)\right) - rV(y_s) = 0 \tag{7}$$

where $H(y_s, V'(y_s)) \equiv \mathrm{Min}_{q\in[q^-,q^+]}\left((nq - c - \lambda - \rho y_s) V'(y_s) + nq\right)$ is the associated Hamiltonian.

Proposition 1. *The optimal premium policy is the Bang–Bang control defined as*

$$q = \begin{cases} q^+ & \text{if } V'(y_s) < -1 \\ \text{undefined} & \text{if } V'(y_s) = -1 \\ q^- & \text{if } V'(y_s) > -1. \end{cases} \tag{8}$$

Proof. Since the Hamiltonian is linear in the control variable, applying Pontryagin's minimum principle leads to the conclusion. □

Thus, under the optimal policy, the premium rate of mutual insurance switches abruptly between the two bounds, and the switching time is determined by the sign of the switching function $G(y_s) \equiv V'(y_s) + 1$. Given this particular structure, one way to solve the optimal premium policy is to find the switching times. However, we will proceed to solve the policy by finding the threshold level of reserves (y_s^*) at which $V'(y_s^*) = -1$. The optimal premium control can thus be determined by comparing initial reserve position with the control threshold.

Proposition 2. *There exists uniquely a reserve level y_s^* at which $V'(y_s^*) = -1$; $V'(y_s) < -1$ if $y_s < y_s^*$, and $V'(y_s) > -1$ if $y_s > y_s^*$.*

Proof. Please see Appendix A. □

To characterize the solution of y_s^*, we let $V^L(y_s)$ and $V^U(y_s)$ denote the expected insurance cost functions when $y_s < y_s^*$ and $y_s > y_s^*$ respectively. By the continuity of the expected insurance cost function, we have $\lim_{y_s\to y_s^*-0} V^L(y_s) = \lim_{y_s\to y_s^*+0} V^U(y_s) = V(y_s^*)$, and because $V'(y_s^*) = -1$ we also have $\lim_{y_s\to y_s^*-0} \frac{dV^L(y_s)}{dy_s} = \lim_{y_s\to y_s^*+0} \frac{dV^U(y_s)}{dy_s} = -1$. In another word, $V^L(y_s)$ meets $V^U(y_s)$ smoothly at y_s^* (continuity and smooth pasting). The solutions of $V^L(y_s)$ and $V^U(y_s)$ can be obtained by solving the ODE in Eq. (7).

Proposition 3. *If $2r = \rho$, the solutions of $V^L(y_s)$ and $V^U(y_s)$ can be expressed in terms of modified Bessel functions of the first kind and the second kind; if $2r \neq \rho$, their solutions can be expressed in terms of the confluent of hypergeometric function of the first kind.*

Proof. Please see Appendix B. □

There are four unknown integration constants in the solutions. The four integration constants and the threshold level of reserves can be determined by the following five conditions:

Two boundary conditions:

$$V^L(0) = K \tag{9}$$

$$\lim_{y_s\to+\infty} V^U(y_s) = \int_s^{+\infty} nq^- e^{-r(t-s)}dt = \frac{nq^-}{r}. \tag{10}$$

Three continuity and smooth pasting conditions:

$$\left.\frac{dV^L(y_s)}{dy_s}\right|_{y_s=y_s^*} = -1 \tag{11}$$

$$\left.\frac{dV^U(y_s)}{dy_s}\right|_{y_s=y_s^*} = -1 \tag{12}$$

$$V^L(y_s^*) = V^U(y_s^*). \tag{13}$$

The first boundary condition states that the mutual insurance firm is ruined instantly and incurs the bankruptcy costs of K if $y_s = 0$; the second boundary condition states that policyholders receive dividends in infinite time horizon if y_s approaches infinity; the three continuity and smooth pasting conditions are those described previously. The details of solving the five unknowns can be found in Appendix C.

Given the solution of y_s^*, we can characterize the optimal premium policy completely. For example, when paying dividends is not allowed such that $q^- = 0$, the optimal policy is to let net reserves drift freely above the threshold, and to collect premium at the maximal rate from policyholders whenever the net reserves drop below the threshold. This premium control can also be understood as to recapitalize the mutual insurance firm once net reserves drop below the threshold. Under this interpretation, the control threshold is the least required capitalization of the firm.

3. Numerical example

The solutions of the ODE in Eq. (7) are expressed in terms of special functions which cannot be evaluated analytically, so we can only solve Eqs. (9)–(13) numerically. In this section, we first use a numerical example to illustrate the solution to the optimal

[6] Basically it means that the constraint in the cost minimization problem in (6) is relaxed.

[7] Intuitively, we can expect that optimal commitment is increasing with K; a larger K implies larger marginal benefits of increasing the commitment.

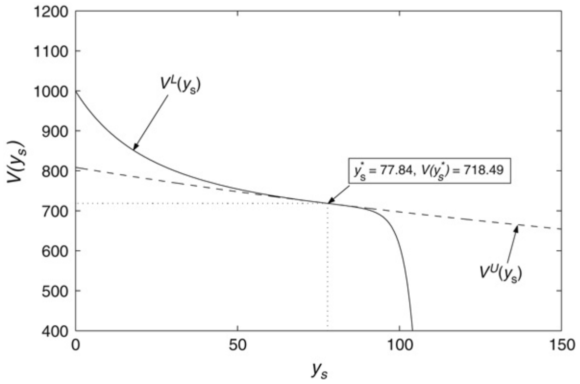

Fig. 1. Solution of the base case.

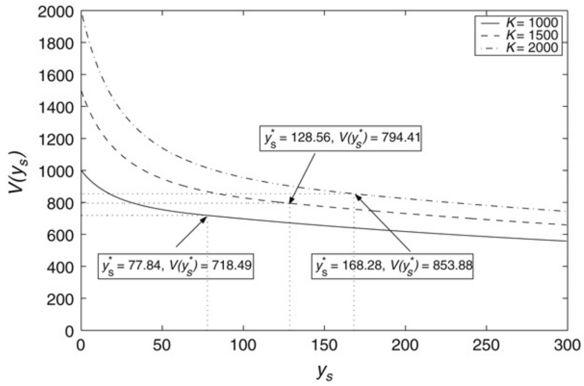

Fig. 2. Expected insurance cost and optimal threshold control under different bankruptcy costs.

premium policy, and then use simulations to investigate how the optimal policy responds to changes of outside factors.[8]

In the base case, we consider a mutual insurance firm with the following setting: $r = 0.10$ (discount rate), $\rho = 0.15$ (unit cost of carrying excessive reserves), $c = 50$ (fixed operation costs), $\lambda = 10$ (arrival rate of aggregated claims), $\sigma^2 = 100$ (variance rate of aggregated claims), and $K = 1000$ (bankruptcy costs). $c + \lambda$ can be interpreted as the expected cost rate of the firm and the fixed operation costs (c) are assumed to be shared by policyholders equally. The nominal premium rate is set as $q^+ = \frac{c+\lambda}{n}$ and we assume that the firm does not pay dividends to policyholders. So in the base case the premium commitment $nq^+ = 60$ and the maximal dividend rate $q^- = 0$. The control threshold of reserves in the base case is found to be 77.84. The firm will then collect the maximal premium of 60 from policyholders when net reserves drop below 77.84, and do nothing when net reserves are above the threshold.

Fig. 1 plots the expected insurance cost function of the base case. The solid line is the plot of $V^L(y_s)$, the part of expected insurance cost function when $y_s < y_s^*$; the dashed line is the plot of $V^U(y_s)$, the part of expected insurance cost function when $y_s > y_s^*$. $V^L(y_s)$ meets $V^U(y_s)$ at the threshold of 77.84 smoothly; their slopes are both -1 at the threshold. When $y_s = 0$, the firm is ruined instantly and incurs the bankruptcy costs of 1000. The expected insurance cost function is strictly decreasing and convex in y_s. As net reserves go to infinity, the expected insurance costs approach zero.

3.1. Comparative statics

Given the nominal premium rate and the dividend policy, the optimal threshold control of the mutual insurance firm is affected by various factors.

Changes of bankruptcy costs

If bankruptcy costs increase, the mutual insurance firm will increase the control threshold of net reserves to reduce the probability of bankruptcy. This can be seen from Fig. 2, which shows three cost curves when bankruptcy costs are 1000 (the base case), 1500, and 2000 respectively. The firm increases the control threshold from 77.84 in the base case to 128.56 when $K = 1500$ and to 168.28 when $K = 2000$. Increasing the control threshold implies that the mutual insurance firm carries larger excessive reserves and thus pays higher operation costs. Due to both larger bankruptcy costs and operation costs, the expected insurance cost curve shifts upward. Since bankruptcy costs represent the costs of the alternative insurance, mutual insurance firms are likely to carry larger excessive reserves and thus pay higher operation costs when outside alternatives are more expensive.

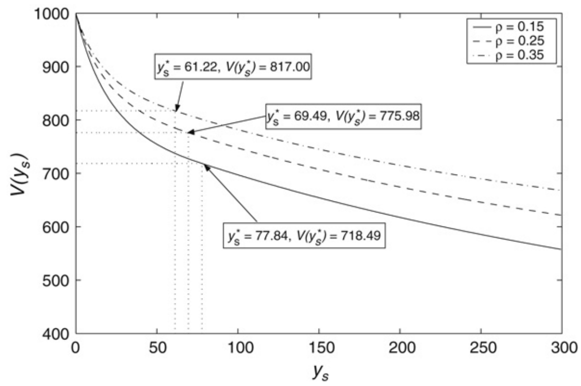

Fig. 3. Expected insurance cost and optimal threshold control under different costs of carrying excessive reserves.

Changes of the unit cost of carrying excessive reserves

The control threshold is lower when the unit cost of carrying excessive reserves is higher. By reducing the control threshold can the firm reduce operation costs, but the trade-off is to increase the probability of bankruptcy. Fig. 3 shows three expected insurance cost curves when ρ is 0.15 (the base case), 0.25, and 0.35 respectively. The control threshold drops to 69.49 from 77.84 of the base case when ρ is increased to 0.25, and drops to 61.22 when ρ is further increased to 0.35. These results suggest that inefficiency in operation can increase the probability of bankruptcy.

Changes of uncertainty

Uncertainty faced by the mutual insurance firm is represented by the variance rate of aggregate claims. A larger variance rate implies larger uncertainty and leads to a higher control threshold for the purpose of protecting against more frequent large claims. As shown in Fig. 4, when the variance rate is increased from 100 of the base case to 200, the control threshold is increased from 77.84 to 87.99; the threshold becomes 98.47 if the variance rate is further increased to 400. These results suggest that mutual insurance firms which are likely to have large claims tend to set high control thresholds and thus carry large excessive reserves. For example, marine mutual insurance firms who may have very large liability costs of accidents normally carry a large amount of excessive reserves.

8 The Bessel functions of the first and second kind can be evaluated using internal functions in MATLAB. As for the confluent hypergeometric function of the first kind in equations, we implement the algorithm in Spanier and Oldham (1987) using MATLAB.

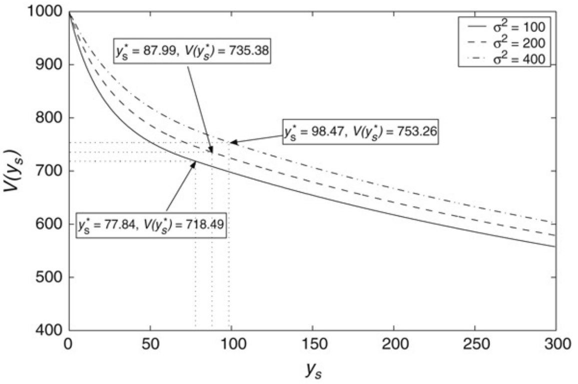

Fig. 4. Expected insurance cost and optimal threshold control under different risk uncertainties.

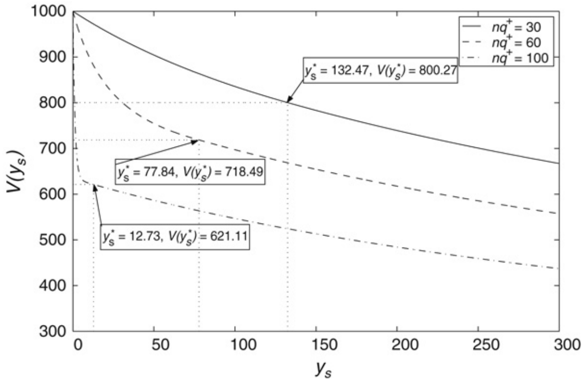

Fig. 6. Expected insurance cost and optimal threshold control under different nominal premium rates.

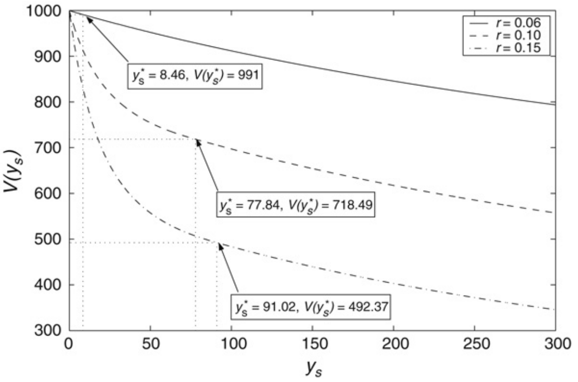

Fig. 5. Expected insurance cost and optimal threshold control under different discount rates.

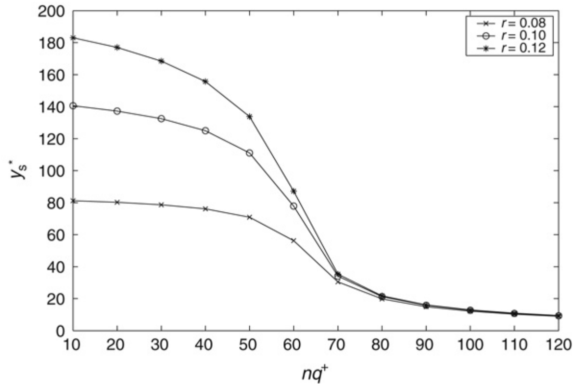

Fig. 7. Relationship between control threshold and premium commitment under different discount rates.

Changes of discount rate

A larger discount rate means that future costs are discounted more by the mutual insurance firm, whose decision is then more myopic by weighting current costs more in dynamic decisions. As a result, the mutual insurance firm will increase the control threshold to reduce the current probability of bankruptcy when the discount rate increases. This can be confirmed by Fig. 5. If the discount rate rises to 0.15 from the base case of 0.10, the control threshold will be raised to 91.02 from 77.84 of the base case. On the other hand, if the discount rate drops to 0.06, the control threshold will be reduced to only 8.46. Since the discount rate is largely based on the interest rate, when the interest rate is higher, we can expect that mutual insurance firms will carry larger excessive reserves.

3.2. *Effects of changing nominal premium rate*

We have investigated how the optimal threshold control responds to the changes of different factors given the nominal premium rate and the dividend policy. In practice, both the nominal premium rate and the dividend policy should also be decisions which are made jointly with the control threshold. As mentioned before, the nominal premium rate can be set lower when the maximal dividend rate is lower (paying fewer dividends); when the nominal premium rate is lower, the mutual insurance firm will rely more on excessive reserves to cover claims and the control threshold will then be set higher. Fig. 6 plots the expected insurance cost curves when the premium commitment (nq^+) is 30, 60 (the base case), and 100 respectively. The control threshold is increased and the cost curve shifts upward when the premium commitment drops. These results suggest that insurance costs of a mutual insurance firm can be reduced by increasing both the nominal premium rate and the dividend rate. However, although insurance costs can be reduced, policyholders' productive activities will be affected negatively by doing so because fewer resources are committed to these activities. The model of this paper focuses only on insurance activities, but it still can generate some insight on, at what circumstances it is more beneficial for a mutual insurance firm to increase the nominal premium rate and the dividend rate.

In Figs. 7 and 8, we plot the relationship between the control threshold and the premium commitment under different discount rates and different unit costs of carrying excessive reserves respectively. The findings can be summarized as:

- In general, increasing the premium commitment makes it possible for the mutual insurance firm to reduce the control threshold to save operation costs.
- The control threshold is more sensitive to the change in the premium commitment when the premium commitment is closer to the expected cost rate of the mutual insurance firm.
- When the premium commitment is not large (compared with the expected cost rate), the control threshold is more sensitive to the change in the premium commitment if the discount rate is higher and the unit cost of carrying excessive reserves is lower.

A higher responsiveness of the control threshold to the change in the nominal premium rate implies larger marginal benefits (from operation cost saving) of increasing the nominal premium rate. Thus, it is more beneficial for a mutual insurance firm to increase its nominal premium rate and dividend rate when:

- the firm's current nominal premium rate is closer to its expected cost rate; and
- the discount rate is higher and the firm's unit cost of carrying excessive reserves is lower.

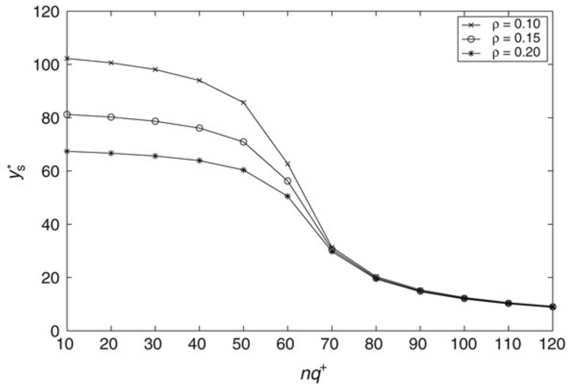

Fig. 8. Relationship between control threshold and premium commitment under different costs of carrying excessive reserves.

4. Conclusion

In this paper, we study the optimal premium policy of mutual insurance given a preset maximal premium rate and a preset maximal dividend rate. The optimal premium policy is found to be the Bang–Bang control which can be fully characterized by a threshold of net reserves along with the two premium bounds. We provide a solution to the optimal policy and investigate how the policy responds to the changes of outside factors. The results of this paper are useful for mutual insurance firms to design premium policies and can be used to test the behavior of these firms in empirical studies.

Acknowledgements

The authors are very grateful for the comments from Alain Bensoussan and the reviewers of the journal. This research was supported in part by The Research Grants Council of Hong Kong under CERG PolyU5230/06E and CERG PolyU5428/05H.

Appendix A. Proof of Proposition 2

The proof includes two parts: the existence and the uniqueness of the threshold.

1. *Existence of* y_s^*

We first show that the case of $V'(y_s) < -1$ for any $y_s \in (0, +\infty)$ cannot exist. If it is true from Eq. (8) we can have

$$\lim_{y_s \to +\infty} V(y_s) = \int_s^{+\infty} nq^+ \mathrm{e}^{-r(t-s)} \mathrm{d}t = \frac{nq^+}{r}. \tag{A.1}$$

(A.1) implies that $V'(y_s) \to 0$ when $y_s \to +\infty$ and this contradicts the claim of $V'(y_s) < -1$ for any $y_s \in (0, +\infty)$. Second, the case of $V'(y_s) > -1$ for any $y_s \in (0, +\infty)$ cannot exist too because the best strategy under the case is to set q^- as low as possible (or to set the dividends as large as possible), and the firm never collects premium but returns all its reserves to policyholders for any y_s. By the definition in Eq. (5), the mutual insurance firm is ruined instantly. So there must be reserves denoted by y_s^* at which $V'(y_s^*) = -1$.

2. *Uniqueness of* y_s^*

This can be obtained by showing that $V(y_s)$ is strictly decreasing and convex in y_s. Considering a time interval $(t, t + \Delta t)$, and we have

$$\begin{aligned} \Pr(\tau \in (t, t+\Delta t)) &= \Pr(y_{t+\Delta t} \le 0 | y_t, q_t) \\ &= 1 - \Phi\left(\frac{y_t + (nq_t - c - \rho y_t - \gamma)\Delta t}{\sigma\sqrt{\Delta t}}\right) \end{aligned} \tag{A.2}$$

where $\Phi(\cdot)$ is the c.d.f. of the standard normal distribution. The expected insurance cost function when $y_s \neq y_s^*$ can be reformulated as follows

$$\begin{aligned} V(y_s) &= E\left(\int_s^{\tau} nq\mathrm{e}^{-r(t-s)}\mathrm{d}t + K\mathrm{e}^{-r(\tau-s)}\right) \\ &= \int_s^{+\infty} \left\{\int_s^t nq\mathrm{e}^{-r(z-s)}dz + K\mathrm{e}^{-r(t-s)}\right\} \cdot \Pr(\tau = t)\,\mathrm{d}t \end{aligned} \tag{A.3}$$

where q takes q^+ or q^- depending on the sign of $V'(y_s) + 1$; $\Pr(\tau = t) = \lim_{\Delta t \to 0} \Pr(\tau \in (t, t + \Delta t))$.

By direct differentiation, it is easy to show that in (A.3), $V'(y_s) < 0$ and $V''(y_s) > 0$ for any $\Delta t > 0$, and for both $q = q^+$ and $q = q^-$. This implies that $V(y_s)$ is strictly decreasing and convex for all $y_s \neq y_s^*$. Consider a y_s^* at which $V'(y_s^*) = -1$, by the continuity and smoothness of $V(y_s)$, we have

$$\lim_{y_s \to y_s^{*+0}} V'(y_s) = \lim_{y_s \to y_s^{*-0}} V'(y_s) = V'(y_s^*) = -1. \tag{A.4}$$

By the strict convexity of $V(y_s)$ for all $y_s \neq y_s^*$, this y_s^* must be unique and $V'(y_s) < -1$ if $y_s < y_s^*$, and $V'(y_s) > -1$ if $y_s > y_s^*$.

Appendix B. General solutions of the HJB equation

Recall that the HJB equation in (3) has the following form

$$\frac{1}{2}\sigma^2 V''(y_s) + \mu(y_s, q) V'(y_s) - rV(y_s) = -nq \tag{B.1}$$

where $\mu(y_s, q) = q - C - \lambda - \rho y_s = \alpha - \rho y_s$. The general solution of the HJB equation can be constructed by any of its particular solutions and the general solution of the corresponding homogeneous equation. Obviously, one particular solution for the HJB equation is nq/r, and the corresponding homogeneous equation of the HJB is

$$\frac{1}{2}\sigma^2 V''(y_s) + \mu(y_s, q) V'(y_s) - rV(y_s) = 0. \tag{B.2}$$

We can make the following transformation

$$\begin{aligned} V(y_s) &= U(y_s)\exp\left(-\frac{1}{2}\int \frac{2\mu(y_s, q)}{\sigma^2}\mathrm{d}y_s\right) \\ &= U(y_s)\exp\left(\frac{\rho}{2\sigma^2}\left(y_s - \frac{\alpha}{\rho}\right)^2 - \frac{\alpha^2}{2\sigma^2\rho}\right). \end{aligned} \tag{B.3}$$

Then $U(y_s)$ is decided by the following differential equation

$$U''(y_s) + f(y_s)U(y_s) = 0 \tag{B.4}$$

where

$$f(y_s) = -\left(\frac{\rho}{\sigma^2}\right)^2 y_s^2 + \frac{2\alpha\rho}{(\sigma^2)^2} y_s - \left(\frac{\alpha}{\sigma^2}\right)^2 + \frac{\rho - 2r}{\sigma^2}.$$

Let $b_1 = \left(\frac{\rho}{\sigma^2}\right)^2$, $b_2 = \frac{-2\alpha\rho}{(\sigma^2)^2}$, and $b_3 = \left(\frac{\alpha}{\sigma^2}\right)^2 - \frac{\rho - 2r}{\sigma^2}$, Eq. (B.4) can be written in the form

$$U''(y_s) - \left(b_1 y_s^2 + b_2 y_s + b_3\right) U(y_s) = 0$$

or

$$U''(y_s) - \left(b_1\left(y_s + \frac{b_2}{2b_1}\right)^2 + b_3 - \frac{b_2^2}{4b_1}\right) U(y_s) = 0.$$

Let $z = y_s + \frac{b_2}{2b_1}$, $c = b_3 - \frac{b_2^2}{4b_1}$, and because $\mathrm{d}z = \mathrm{d}y_s$, we can have

$$U''(z) - \left(b_1 z^2 + c\right) U(z) = 0. \tag{B.5}$$

$$A_2 = \frac{-1}{\exp\left(\frac{\rho\underline{\theta}^{*2}}{2\sigma^2} - \frac{\bar{\alpha}^2}{2\sigma^2\rho}\right)\left[\left(\frac{\rho\underline{\theta}^*}{\sigma^2} - \frac{\rho}{\sigma^2}\sqrt{\underline{\theta}^{*2}}\right)D_{-\frac{r}{\rho}}\left(\sqrt{\frac{2\rho\underline{\theta}^{*2}}{\sigma^2}}\right) - \frac{\sqrt{2\rho}r}{\sigma\rho}D_{-\frac{r+\rho}{\rho}}\left(\sqrt{\frac{2\rho\underline{\theta}^{*2}}{\sigma^2}}\right)\right]}$$

Box I.

If $c = 0$, which implies that $2r = \rho$, the solution of (B.5) is expressed in terms of modified Bessel functions of the first kind $I_v(x)$, and second kind $K_v(x)$ (Polyanin and Zaitsev, 1995, p. 132).

$$U(z) = \sqrt{z}\left[AI_{\frac{1}{4}}\left(\frac{\sqrt{b_1}}{2}z^2\right) + BK_{\frac{1}{4}}\left(\frac{\sqrt{b_1}}{2}z^2\right)\right] \tag{B.6}$$

where A and B are two integration constants. Transforming the variables back and applying (B.3), we get the solutions as

$$V^L(y_s) = \exp\left(\frac{\rho}{2\sigma^2}\bar{\theta}^2 - \frac{\bar{\alpha}^2}{2\sigma^2\rho}\right) \times \frac{\rho}{\sigma^2}\left[A_1 I_{\frac{1}{4}}\left(\frac{\rho}{2\sigma^2}\bar{\theta}^2\right) + B_1 K_{\frac{1}{4}}\left(\frac{\rho}{2\sigma^2}\bar{\theta}^2\right)\right] + \frac{2nq^+}{\rho} \tag{B.7}$$

$$V^U(y_s) = \exp\left(\frac{\rho}{2\sigma^2}\underline{\theta}^2 - \frac{\underline{\alpha}^2}{2\sigma^2\rho}\right) \times \frac{\rho}{\sigma^2}\left[A_2 I_{\frac{1}{4}}\left(\frac{\rho}{2\sigma^2}\underline{\theta}^2\right) + B_2 K_{\frac{1}{4}}\left(\frac{\rho}{2\sigma^2}\underline{\theta}^2\right)\right] + \frac{2nq^-}{\rho} \tag{B.8}$$

where $\bar{\theta} = y_s - \frac{q^+ - C - \lambda}{\rho} = y_s - \frac{\bar{\alpha}}{\rho}$, and $\underline{\theta} = y_s - \frac{q^- - C - \lambda}{\rho} = y_s - \frac{\underline{\alpha}}{\rho}$.

If $c \neq 0$, we can further transform (B.5) as the following standard form

$$U''(x) - \left(\frac{1}{4}x^2 + \frac{c}{2\sqrt{b_1}}\right)U(x) = 0. \tag{B.9}$$

The general solution of (B.9) can be found in Abramowitz and Stegun (1972, p. 686), and the general solutions of the HJB equation can then be finally expressed as

$$V^L(y_s) = \exp\left(\frac{-\bar{\alpha}^2}{2\sigma^2\rho}\right)\left[A_1 M\left(\frac{r}{2\rho}, \frac{1}{2}; \frac{\rho}{\sigma^2}\bar{\theta}^2\right) + B_1\frac{\sqrt{\rho}}{\sigma}\bar{\theta}M\left(\frac{r+\rho}{2\rho}, \frac{3}{2}; \frac{\rho}{\sigma^2}\bar{\theta}^2\right)\right] + \frac{nq^+}{r} \tag{B.10}$$

$$V^U(y_s) = \exp\left(\frac{-\underline{\alpha}^2}{2\sigma^2\rho}\right)\left[A_2 M\left(\frac{r}{2\rho}, \frac{1}{2}; \frac{\rho}{\sigma^2}\underline{\theta}^2\right) + B_2\frac{\sqrt{\rho}}{\sigma}\underline{\theta}M\left(\frac{r+\rho}{2\rho}, \frac{3}{2}; \frac{\rho}{\sigma^2}\underline{\theta}^2\right)\right] + \frac{nq^-}{r} \tag{B.11}$$

where $M(a, b; x)$ represents the confluent hypergeometric function of the first kind with two parameters a and b. Based on the interrelationship between the confluent hypergeometric function of the first kind and the parabolic cylinder function, the solution in (B.10) and (B.11) can be equivalently expressed in terms of the parabolic cylinder function $D_v(x)$

$$V^L(y_s) = \exp\left(\frac{\rho}{2\sigma^2}\bar{\theta}^2 - \frac{\bar{\alpha}^2}{2\sigma^2\rho}\right) \times \left[AD_{-\frac{r}{\rho}}\left(\sqrt{\frac{2\rho}{\sigma^2}}\bar{\theta}^2\right) + BD_{-\frac{r}{\rho}}\left(-\sqrt{\frac{2\rho}{\sigma^2}}\bar{\theta}^2\right)\right] + \frac{nq^+}{r} \tag{B.12}$$

$$V^U(y_s) = \exp\left(\frac{\rho}{2\sigma^2}\underline{\theta}^2 - \frac{\underline{\alpha}^2}{2\sigma^2\rho}\right) \times \left[AD_{-\frac{r}{\rho}}\left(\sqrt{\frac{2\rho}{\sigma^2}}\underline{\theta}^2\right) + BD_{-\frac{r}{\rho}}\left(-\sqrt{\frac{2\rho}{\sigma^2}}\underline{\theta}^2\right)\right] + \frac{nq^-}{r}. \tag{B.13}$$

Appendix C. Decide the unknowns in the general solution of HJB equation

From Eqs. (B.12) and (B.13), by the conditions of $V^L(0) = K$, and $\frac{\mathrm{d}V^L(y_s)}{\mathrm{d}y_s}\big|_{y_s=y_s^*} = -1$, we can decide the two integration constants in $V^L(y_s)$ as the functions of y_s^*. Specifically, they are

$$A_1 = \frac{K - \frac{n\bar{q}}{r}}{D_{-\frac{r}{\rho}}\left(\sqrt{\frac{2\bar{\alpha}^2}{\sigma^2\rho}}\right)} - \frac{D_{-\frac{r}{\rho}}\left(-\sqrt{\frac{2\bar{\alpha}^2}{\sigma^2\rho}}\right)}{D_{-\frac{r}{\rho}}\left(\sqrt{\frac{2\bar{\alpha}^2}{\sigma^2\rho}}\right)}B_1 = \phi - \gamma B_1 \tag{C.1}$$

$$B_1 = \frac{-\exp\left(\frac{\bar{\alpha}^2}{2\sigma^2\rho} - \frac{\rho}{2\sigma^2}\bar{\theta}^{*2}\right) - \phi W(y_s^*)}{F(y_s^*) - \gamma W(y_s^*)} \tag{C.2}$$

with

$$W(y_s^*) = \left(\frac{\rho\bar{\theta}^*}{\sigma^2} - \frac{\rho}{\sigma^2}\sqrt{\bar{\theta}^{*2}}\right)D_{-\frac{r}{\rho}} \times \left(\sqrt{\frac{2\rho\bar{\theta}^{*2}}{\sigma^2}}\right) - \frac{\sqrt{2\rho}r}{\sigma\rho}D_{-\frac{r+\rho}{\rho}}\left(\sqrt{\frac{2\rho\bar{\theta}^{*2}}{\sigma^2}}\right)$$

$$F(y_s^*) = \left(\frac{\rho\bar{\theta}^*}{\sigma^2} - \frac{\rho}{\sigma^2}\sqrt{\bar{\theta}^{*2}}\right)D_{-\frac{r}{\rho}} \times \left(-\sqrt{\frac{2\rho\bar{\theta}^{*2}}{\sigma^2}}\right) + \frac{\sqrt{2\rho}r}{\sigma\rho}D_{-\frac{r+\rho}{\rho}}\left(-\sqrt{\frac{2\rho\bar{\theta}^{*2}}{\sigma^2}}\right)$$

where $\bar{\alpha} = \bar{q} - C - \lambda$; $\bar{\theta}^*$ represents $\bar{\theta}$ evaluated at y_s^*.

The convergence of expected insurance cost requires that as $y_s \to +\infty$, $V^U(y_s) \to \frac{n\underline{q}}{r}$. We must set $B_2 = 0$, because $\lim_{x\to-\infty} D_v(x) = +\infty$ when $v < 0$. Also, for the large reserve, the other part in the solution can be approximated as

$$A_2\exp\left(-\frac{\underline{\alpha}^2}{2\sigma^2\rho}\right)\left(\sqrt{\frac{2\rho\underline{\theta}^2}{\sigma^2}}\right)^{\frac{-r}{\rho}}\left[1 - \left(\frac{r}{\rho} + \frac{r^2}{\rho^2}\right)\left(\frac{4\rho\underline{\theta}^2}{\sigma^2}\right)^{-2}\right]$$

which converges to zero as $y_s \to +\infty$. Similarly, $\underline{\alpha} = \underline{q} - C - \lambda$; $\underline{\theta}^*$ represents $\underline{\theta}$ evaluated at y_s^*. A_2 can be decided from the condition of $\frac{\mathrm{d}V^U(y_s)}{\mathrm{d}y_s}\big|_{y_s=y_s^*} = -1$. Specifically, it is given by the equation in Box I. Finally, by the value matching requirement of $V^L(y_s^*) = V^U(y_s^*)$, we can solve the ODE completely, but the solutions can only be obtained numerically.

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