EconS 510

Takehome Quiz #3 Answer Key

Problem 1.

a) 1 year = 8,760 hours

$$P(x \le 8.76) = \int_0^{8.76} 0.01e^{-x/100}dx = -e^{-x/100}\Big|_0^{8.76} = -e^{-0.0876} + 1 = 0.0839$$

b) $P(x \ge 100) = 1 - P(x \le 100) = 1 - \int_0^{100} 0.01e^{-x/100}dx = 1 + e^{-x/100}\Big|_0^{100} = e^{-1} = 0.3679$

c) $P(x \ge 100 \mid x \ge 50) = \dfrac{P(x \ge 100)}{P(x \ge 50)} = \dfrac{e^{-1}}{e^{-.5}} = 0.6065$

d) $P(x \ge 150 \mid x \ge 50) = \dfrac{P(x \ge 150)}{P(x \ge 50)} = \dfrac{e^{-1.5}}{e^{-.5}} = 0.3679$

Problem 2

a) $f(x_1, x_2) = \int_0^1 \left(\dfrac{8}{3}\right)[x_1 + 2x_2]x_3^3 dx_3 = \left(\dfrac{2}{3}\right)[x_1 + 2x_2]x_3^4\Big|_0^1 = \left(\dfrac{2}{3}\right)[x_1 + 2x_2]I_{[0,1]}(x_1)I_{[0,1]}(x_2)$

$$P(x_1 > .75, x_2 > .75) = \int_{0.75}^1 \int_{0.75}^1 \left(\frac{2}{3}\right)[x_1 + 2x_2]dx_2 dx_1$$

$$= \left(\frac{2}{3}\right)\int_{0.75}^1 (0.25x_1 + 0.4375)dx_1 = \left(\frac{2}{3}\right)\left(\frac{0.25}{2}x_1^2 + 0.4375x_1\right)\Bigg|_{0.75}^1 = \frac{7}{64} = 0.1094$$

b) $f(x_3) = \int_0^1 \int_0^1 \left(\dfrac{8}{3}\right)[x_1 + 2x_2]x_3^3 dx_2 dx_1 = \left(\dfrac{8}{3}\right)x_3^3 \int_0^1 \left[\dfrac{1}{2}x_1^2 + x_1\right]dx_1 = 4x_3^3 I_{[0,1]}(x_3)$

$$P(x_3 > .8) = \int_{0.8}^1 4x_3^3 dx_3 = x_3^4\Big|_{0.8}^1 = 0.5904$$

c) $f(x_1) = \int_0^1 \left(\dfrac{2}{3}\right)[x_1 + 2x_2]dx_2 = \left(\dfrac{2}{3}\right)[x_1 + 1]I_{[0,1]}(x_1)$

$$f(x_2 \mid x_1 > 0.75) = \frac{f(x_2, x_1 > 0.75)}{f(x_1 > 0.75)} = \frac{\int_{0.75}^{1} \left(\frac{2}{3}\right)[x_1 + 2x_2]dx_1}{\int_{0.75}^{1} \left(\frac{2}{3}\right)[x_1 + 1]dx_1} = \frac{7+16x_2}{15} I_{[0,1]}(x_2)$$

$$P(x_2 > 0.75 \mid x_1 > 0.75) = \int_{0.75}^{1} \frac{7+16x_2}{15} dx_2 = \frac{7}{20} = 0.35$$

Problem 3

a) $P(pq < 2) = \int_{3.5}^{4} \int_{0}^{2/p} 2pe^{-pq} dq dp = \int_{3.5}^{4} \left(2 - 2e^{-2}\right) dp = 1 - e^{-2} = 0.8645$

b) $f(p) = \int_{0}^{\infty} 2pe^{-pq} dq = 2I_{[3.5,4]}(p)$

$P(p > 3.75) = \int_{3.75}^{4} 2dp = 0.5$

c) $f(q \mid p = 3.75) = 3.75e^{-3.75q} I_{(0,\infty)}(q)$

$P(q > 10 \mid p = 3.75) = \int_{10}^{\infty} 3.75e^{-3.75q} dq = -e^{-3.75q}\Big|_{10}^{\infty} = e^{-37.5}$

d) $f(q \mid p = 4) = 4e^{-4q} I_{(0,\infty)}(q)$

$P(q > 10 \mid p = 4) = \int_{10}^{\infty} 4e^{-4q} dq = -e^{-4q}\Big|_{10}^{\infty} = e^{-40}$

The results make economic sense, because the higher price results in a lower probability of selling the same amount of gas.

Problem 4

a) $F(x, y) = \begin{cases} 0 & x, y < 0 \\ x^3 y^2 & x, y \in [0,1] \\ x^3 & x \in [0,1], y > 1 \\ y^2 & y \in [0,1], x > 1 \\ 1 & x, y > 1 \end{cases}$

$P(x \le .5, y \le .75) = 0.5^3 * 0.75^2 = 0.0703$

b) $F(x)=\begin{cases} 0 & x<0 \\ x^3 & x\in[0,1] \\ 1 & x>1 \end{cases}$

$P(x\le .5)=0.125$

c) $f(x)=F'(x)=3x^2 I_{[0,1]}(x)$

Problem 5

a) Since the number of disc players is discrete and not infinitely divisible, the total production and the proportions cannot take any possible value in the set of real numbers.

b)

$$f(x<1,y<0.5,z<0.5)=\int_0^1\int_0^{0.5}\int_0^{0.5}\frac{3}{7}\left[x+y^2+2z\right]e^{-x}dzdydx=\int_0^1\int_0^{0.5}\frac{3}{7}\left[\frac{1}{2}x+\frac{1}{2}y^2+\frac{1}{4}\right]e^{-x}dydx$$
$$=\int_0^1\frac{3}{28}xe^{-x}dx+\int_0^1\left[\frac{1}{16}e^{-x}\right]dx=-\frac{3}{28}\int_0^1 xde^{-x}+\frac{1}{16}\left(1-e^{-1}\right)$$
$$=-\frac{3}{28}\left(xe^{-x}\Big|_0^1-\int_0^1 e^{-x}dx\right)+\frac{1}{16}\left(1-e^{-1}\right)=-\frac{6e^{-1}}{28}-\frac{e^{-1}}{16}+\frac{3}{28}+\frac{1}{16}=0.0678$$

c) $f_x(x)=\int_0^1\int_0^1\frac{3}{7}\left[x+y^2+2z\right]e^{-x}dzdy=\int_0^1\frac{3}{7}\left[x+y^2+1\right]e^{-x}dy=\left[\frac{3}{7}x+\frac{4}{7}\right]e^{-x}I_{[0,\infty)}(x)$

$P(x<1)=\int_0^1\left[\frac{3}{7}x+\frac{4}{7}\right]e^{-x}dx=1-\frac{10}{7}e^{-1}=0.4745$

d)

$$f_{yz}(y,z)=\int_0^\infty\frac{3}{7}\left[x+y^2+2z\right]e^{-x}dx=\frac{3}{7}\int_0^\infty\left[xe^{-x}+\left[y^2+2z\right]e^{-x}\right]dx$$
$$=\frac{3}{7}\left[y^2+2z+1\right]I_{[0,1]}(y)I_{[0,1]}(z)$$

$$P(y<.5,z<.5)=\int_0^{0.5}\int_0^{0.5}\frac{3}{7}\left[y^2+2z+1\right]dzdy=\int_0^{0.5}\frac{3}{7}\left[y^2\frac{1}{2}+\frac{3}{4}\right]dy=\frac{3}{7}\frac{19}{48}=\frac{19}{112}=0.1696$$

e) $$f(x\mid y=0.5)=\frac{\int_0^1\frac{3}{7}\left[x+\frac{1}{4}+2z\right]e^{-x}dz}{f(y=0.5)}=\frac{\frac{3}{7}\left[x+\frac{5}{4}\right]e^{-x}}{\frac{3}{7}\left[\frac{1}{4}+2\right]}=\frac{4}{9}\left[x+\frac{5}{4}\right]e^{-x}I_{[0,\infty)}(x)$$

$$f\left(x \le 1 \mid y = 0.5\right) = \int_0^1 \frac{4}{9}\left[x + \frac{5}{4}\right]e^{-x}dx = 1 - \frac{13e^{-1}}{9} = 0.4686$$

Problem 6

a) $f_{x_3}\left(x_3\right) = \sum_{x_1}\sum_{x_2}\left(\frac{1}{576}\right)\left(x_1 + 2x_2 + 3x_3\right) = \sum_{x_1}\left(\frac{1}{576}\right)\left(4x_1 + 12 + 12x_3\right) = \left(\frac{1}{24}\right)\left(3 + 2x_3\right)I_{\{0,1,2,3\}}\left(x_3\right)$

$$P\left(x_3 = 3\right) = \frac{3}{8}$$

b) $f_{x_1x_2}\left(x_1, x_2\right) = \sum_{x_3}\left(\frac{1}{576}\right)\left(x_1 + 2x_2 + 3x_3\right) = \left(\frac{1}{288}\right)\left(2x_1 + 4x_2 + 9\right)I_{\{0,1,2,3\}}\left(x_1\right)I_{\{0,1,2,3\}}\left(x_2\right)$

$$P\left(x_1 > 1, x_2 > 1\right) = \sum_{x_1=2,3}\sum_{x_2=2,3}\left(\frac{1}{288}\right)\left(2x_1 + 4x_2 + 9\right) = \sum_{x_1=2,3}\left(\frac{1}{288}\right)\left(4x_1 + 20 + 18\right) = \frac{96}{288} = 0.3333$$

c) $f\left(x_1 \mid x_2 = 2\right) = \left(\frac{1}{80}\right)\left(2x_1 + 17\right)I_{\{0,1,2,3\}}\left(x_1\right)$

$$P\left(x_1 \le 1 \mid x_2 \ge 2\right) = \frac{P\left(x_1 \le 1, x_2 \ge 2\right)}{P\left(x_2 \ge 2\right)} = \frac{\sum_{x_1=0,1}\sum_{x_2=2,3}\left(\frac{1}{288}\right)\left(2x_1 + 4x_2 + 9\right)}{\sum_{x_2=2,3}\sum_{x_1}\sum_{x_3}\left(\frac{1}{576}\right)\left(x_1 + 2x_2 + 3x_3\right)} = \frac{160}{352} = 0.4545$$

d) $f\left(x_1, x_2 \mid x_3 = 0\right) = \frac{\left(\frac{1}{576}\right)\left(x_1 + 2x_2\right)}{\left(\frac{3}{24}\right)} = \left(\frac{1}{72}\right)\left(x_1 + 2x_2\right)I_{\{0,1,2,3\}}\left(x_1\right)I_{\{0,1,2,3\}}\left(x_2\right)$

$$P\left(x_1 > 1, x_2 > 1, x_3 = 0\right) = \frac{30}{72} = 0.4167$$

e) $\pi = 20x_1 + 30x_2 + 50x_3 - 150$ is the random variable that represents the daily profit.

$$f\left(\pi\right) = \sum_{\pi = 20x_1 + 30x_2 + 50x_3 - 150, (x_1, x_2, x_3) \in \{0,1,2,3\}}\left(\frac{1}{576}\right)\left(x_1 + 2x_2 + 3x_3\right)$$

$$
\begin{aligned}
P(\pi > 0) = &\sum_{x_3=3, x_1 \in \{0,1,2,3\}, x_2 \in \{1,2,3\}} \left(\frac{1}{576}\right)(x_1 + 2x_2 + 3x_3) + \sum_{x_3=3, x_1 \in \{1,2,3\}, x_2 = 0} \left(\frac{1}{576}\right)(x_1 + 2x_2 + 3x_3) \\
&+ \sum_{x_3=2, x_1 \in \{2,3\}, x_2 \in \{1,2,3\}} \left(\frac{1}{576}\right)(x_1 + 2x_2 + 3x_3) + \sum_{x_3=2, x_1 \in \{0,1\}, x_2 \in \{2,3\}} \left(\frac{1}{576}\right)(x_1 + 2x_2 + 3x_3) \\
&+ \sum_{x_3=1, x_1=3, x_2 \in \{2,3\}} \left(\frac{1}{576}\right)(x_1 + 2x_2 + 3x_3) + \sum_{x_3=1, x_1=\{1,2\}, x_2=3} \left(\frac{1}{576}\right)(x_1 + 2x_2 + 3x_3) \\
&= 0.66
\end{aligned}
$$