Duality Between CDFs and PDFs in Multivariate Case (From MSFEB, 2^{nd} edition)

Similar to the univariate case, the joint CDF can be used to derive joint discrete and continuous probability densities. For the discrete case, we discuss the result for bivariate random variables only. For multivariate random variables of three dimensions or higher, the large number of terms required in the density-defining procedure makes its use somewhat cumbersome. We state the generalization in a footnote 1.

Theorem 2.4 ***Discrete Bivariate PDFs from Bivariate CDFs*** *Let* (X, Y) *be a discrete bivariate random variable with joint cumulative distribution function* $F(x,y)$, *and let* $x_1 < x_2 < x_3 \ldots$, *and* $y_1 < y_2 < y_3 < \ldots$, represent the possible outcomes *of X and Y. Then*

a. $f(x_1, y_1) = F(x_1, y_1)$;

b. $f(x_1, y_j) = F(x_1, y_j) - F(x_1, y_{j-1}),\ j \geq 2$;

c. $f(x_i, y_1) = F(x_i, y_1) - F(x_{i-1}, y_1),\ i \geq 2$; and

d. $f(x_i, y_j) = F(x_i, y_j) - F(x_i, y_{j-1}) - F(x_{i-1}, y_j) + F(x_{i-1}, y_{j-1})$, i and $j \geq 2$.

Proof The proof is left to the reader.■

As we remarked in the univariate case, if the range of the random variable is such that a lowest ordered outcome does not exist, then the definition simplifies to $f(x_i, y_j)=F(x_i, y_j)-F(x_i, y_{j-1})-F(x_{i-1}, y_j)+F(x_{i-1}, y_{j-1})$, $\forall$ i and j.

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1 In the discrete m-dimensional case, the PDF can be defined as $f(\mathbf{x}) = F(\mathbf{x}) + \lim_{\delta \to 0_+} \left(\sum_{i=1}^{m} (-1)^i \sum_{\mathbf{v} \in S_i} F(\mathbf{x} - \delta \mathbf{v}) \right)$ where S_i is the set of all of the different $(m \times 1)$ vectors that can be constructed using i 1's and m-i 0's.