1. A mining company has initial reserves of $S_0$ of some mineral which it can extract at a cost of $C(R, S) = a + bR^2 - dS$, where $R$ is the extraction rate, and sell on the world market at a constant price $p$. The government sets a pollution tax, $t$, for every unit of pollution produced by the firm. Total pollution at a given time is equal to $\delta R$ where $\delta$ is a conversion parameter.

(a) Setup the company’s objective function, constraints and Hamiltonian. Solve for the maximum principle, portfolio balance condition and dynamic constraint assuming a discount rate of $r$ and a marginal shadow value of the stock equal to $\lambda$. Interpret each of these conditions.

(b) Derive the optimal extraction rate $\dot{R}$. Using this formula along with the dynamic constraint, derive the slope of the optimal trajectory path and draw it in $(S, R)$ space.

(c) Show that the optimal $R(T)$ is the rate of extraction that minimizes average cost.

(d) If the pollution tax is raised, what will happen to $T$? Explain intuitively why.

(a) 
\[
\max_{R[t] \geq 0} \int_{t=0}^{T} e^{-rt} \left( (p - t\delta)R - (a + bR^2 - dS) \right) dt \quad s.t. \quad \dot{S} = -R, S_0 \text{ given}
\]

\[
H = (p - t\delta)R - (a + bR^2 - dS) - \mu R
\]

MP) $(p - t\delta) - 2bR - \mu = 0$

Price is equal to the pollution tax (adjusted in output terms) plus the marginal cost of extraction and the shadow price of the resource stock.

PB) $\dot{\mu} = r\mu - d$

Growth rate of the shadow price of the resource stock is equal to the opportunity cost of investment ($r$) net of the marginal cost of the stock.

DC) $\dot{S} = -R$

Stock declines by the rate of extraction.

(b) 
Take time derivative of MP, $-2b\dot{R} - \dot{\mu} = 0$. Simplifying yields,

\[
\dot{R} = \frac{d - r((p - t\delta) - 2bR)}{2b}
\]

Since $\dot{S} = -R$, the optimal trajectory has the slope,
\[
\frac{dR}{dS} = \frac{\dot{R}}{\dot{S}} = \frac{r(p - t\delta) - 2bR - d}{2bR} = \frac{r(p - t\delta) - d}{2bR} - r
\]

If \(d\) (the marginal cost from stock) is large such that \(d > r((p - t\delta))\), then the slope is negative. The trajectory moves from right to left and may look like this,

\[ R \]

\[ S \]

It may also be upward sloping if \(d\) is small and \(r\) is also small. Note, because the stock effect is constant, it cannot be U-shaped or have an inverted U-shape. However, it could be negatively sloped at high levels of \(R\) and positively sloped as \(R\) declines.

(c)

One transversality condition is when

\[ H(T) = (p - t\delta)R(T) - (a + bR(T)^2 - dS(T)) - \mu(T)R(T) = 0 \]

Also, from MP, \((p - t\delta) - 2bR(T) - \mu(T) = 0\). Multiplying both sides by \(R(T)\) yields,

\[ (p - t\delta)R(T) - 2bR(T)^2 - \mu(T)R(T) = 0 \]

From both equations we have, \((p - t\delta)R(T) - \mu(T)R(T) = (a + bR(T)^2 - dS(T))\) and \((p - t\delta)R(T) - \mu(T)R(T) = 2bR(T)^2\) which simplifies to \((a + bR(T)^2 - dS(T)) = 2bR(T)^2\). Dividing both sides by \(R(T)\) yields,

\[ \frac{(a + bR(T)^2 - dS(T))}{R(T)} = 2bR(T) \]

Left hand side is average cost and right hand side is marginal cost. This is where average cost is minimized.

(d)

The effect is ambiguous. First there is less extraction per unit of time leading to a longer \(T\) but since the transversality condition \((H(T)=0)\), is also affected where \(S(T)\) decreases, \(T\) is also shorter. The net effect is ambiguous.
2. Suppose that a fish stock has the natural growth curve,
\[ G(S) = (1-S) \]
where \( S \) is stock and the carrying capacity is normalized to 1. The fishery production follows the function,
\[ X = bES \]
where \( b \) is the harvesting efficiency parameter, \( X \) is fish harvested and \( E \) is effort. Suppose the dynamics of effort are given as,
\[ \dot{E} = v[(p - t)bS - w] \]
where \( v \) is the sensitivity parameter of effort, \( p \) is the price of fish per kilogram, \( t \) is an output tax and \( w \) is the constant average cost of effort.

(a) Calculate the open access equilibrium levels of fish stock and effort for the fishery.

Here, \( \dot{E} = 0 \) means \( v[(p - t)bS - w] = 0 \). This results in \( S^{OA} = \frac{w}{(p-t)b} \).

Also, \( \dot{S} = (1-S) - bES = 0 \). This results in \( E^{OA} = \frac{1-s}{bs} = \frac{1}{bS} - \frac{1}{b} = \frac{p-t}{w} - \frac{1}{b} \).

(b) Sketch the phase plane diagram which describes qualitatively the dynamics of open access fishery. Include in your diagram arrows of motion across the isoclines or arrows of motion in each isosector. Intuitively explain the behavior of effort and stock over time.

![Phase Plane Diagram]

Intuition:
With large stock, profit increases. This signals an incentive for an increase in effort. More effort reduces stock. As stock is reduced, profit declines which signals a decline in effort. When effort declines, stock increases again. The cycle occurs again.

(c) Use the linearization technique to show stability properties of the node. (Hint: If \( \text{Det}[A] < 0 \) -- saddle path; \( \text{Det}[A] > 0 \) and \( r_1 \) & \( r_2 < 0 \) -- stable; \( \text{Det}[A] > 0 \) and \( r_1 \) & \( r_2 > 0 \) -- unstable; \( \text{Det}[A] > 0 \), \( r_1 \) & \( r_2 \) complex and \( \text{tr}[A] < 0 \) -- stable focus; \( \text{Det}[A] > 0 \), \( r_1 \) & \( r_2 \) complex and \( \text{tr}[A] > 0 \) -- unstable focus; \( \text{Det}[A] > 0 \), \( r_1 \) & \( r_2 \) complex and \( \text{tr}[A] = 0 \) -- center.)

\[ A = \begin{bmatrix} -1 - bE & -bS \\ v(p-t)b & 0 \end{bmatrix} \]

\( \text{Det}(A) = v(p-t)bbS > 0 \). Note trace is \( -1 - bE < 0 \). Therefore it is a stable focus.
(c) If the regulator deems that there is too much harvest relative to the socially optimal level, what should the tax, \( t \), be equal to? Describe how the regulator should adjust this level at each point in time.

Tax should be equal to marginal user cost of the stock or shadow price of the stock. As stock is large, tax should be low but when stock is low, tax is high.

3. Let \( V(t) \) be the volume function of the tree stand, \( r \) is the discount rate, \( p \) is a constant price from harvesting timber, \( h \) is a constant price per unit cost of harvested timber and \( c \) is a planting cost. Assume that the forest stand produces tree branches that can be sold for fuel at each time period, \( F(t) \), at a price of \( q \). The government sets a tax \( v \) for every unit of tree branches sold to internalize the effect of pollution from burning the fuel source. The landowner plans for only a single rotation where there is no replanting after harvest and the land is left idle.

(a) Set up the landowner’s problem and derive the optimal condition that yields the optimal rotation interval that maximizes timber and non-timber benefits. Give an economic interpretation of this condition.

\[
\max_T (p - h)V(T)e^{-rt} + \int_0^T (q - v)F(t)e^{-rt}dt - c
\]

FOC:

\[
(p - h)\dot{V} + (q - v)F(T) = r(p - h)V
\]

Net marginal value of timber plus net marginal fuel value equals marginal opportunity cost of timber.

(b) Prove that this rotation interval is longer compared to the case that the owner does not internalize any of the fuel value.

The FOC would be

\[
(p - h)\dot{V} - (p - h)V = 0
\]

Recall that with the fuel value we would obtain,

\[
(p - h)\dot{V} - (p - h)V = -(q - v)F(T) < 0
\]

This means there is less marginal benefit from keeping the forest so the rotation length without non-wood value is shorter.

(c) Suppose that tax rate increases. What is the effect of this change compared to the rotation you derived in (a)? Explain why you derive such a result. Provide a simple mathematical or graphical proof of your result.

We need to solve the comparative static,

\[
\frac{dT}{dv} = \frac{-F(T)}{(p - h)\dot{V} + (q - v)F(T) - r(p - h)V}
\]

The denominator is negative if second order condition holds. Since numerator is negative we have, \( \frac{dT}{dv} < 0 \). An increase in tax decreases \( T \). Intuitively, the net marginal benefits from fuel wood are lower which leads to a lower \( T \).