WSU Economics PhD Math Bootcamp 2017

Final Exam

Problem 1.

You are studying a game theoretic problem in which three firms are competing with each other in quantity supplied to a new market. Some firms benefit when other firms produce more and some firms are hurt when others produce more. The problem is complex, but you have finally managed to create the payoff functions for each player (firm) in this game that correctly specify their preferences over outcomes. These payoff functions are shown below.

$$\pi_1(q_1|q_2, q_3) = \frac{1}{4}q_1^2 - q_1q_2 + \frac{1}{3}q_1q_3$$
$$\pi_2(q_2|q_1, q_3) = \frac{1}{4}\left(q_1 - q_2^2\right) + q_2q_3$$
$$\pi_3(q_3|q_1, q_2) = \frac{1}{8}q_3(q_1 + q_2) - q_3$$

You are looking for a **Nash Equilibrium** of the game which requires each firm to choose their quantity q_i to maximize their payoff given the choices of q by the other two firms. Suppose that you have determined that *first order conditions* are sufficient for a maximum in each firm's payoff function (meaning that q_i is chosen to make $\partial\pi_i/\partial q_i = 0$.)

(A) Derive the system of first order conditions for the payoff functions in this game.

(B) Using the set of first order conditions from part (A), rearrange them into a linear system of equations of the form, $A\mathbf{x} = \mathbf{b}$.

(C) An equilibrium will be a list of strategies (q_1, q_2, q_3) in which all three first order conditions are simultaneously satisfied. Perform a test (calculate a special number) on the matrix A from part (B) that will tell you whether or not a unique equilibrium exists for this game (given the vector $\mathbf{b}$).

(D) As a matrix, A represents a linear mapping between which two vector spaces? $A : \mathbb{R}^? \to \mathbb{R}^?$.

(E) Given the value of the number calculated in part (C), what is the rank of the matrix A? What is the dimension of its null space?

(F) *If possible*, find the inverse of the matrix A and use it to calculate the equilibrium values of (q_1, q_2, q_3) (i.e., those that solve the system). If it is not possible, explain why and how you know it.

Problem 2.

Consider an investor who must decide how much money, y, to invest in a stock to maximize their expected utility. Suppose that there are only two possible outcomes G for good and B for bad. If the consumer believes the probability of G is p_G and the probability of B is $1 - p_G$, then the expected utility from a choice $y > 0$ is,

$$p_G U(y; G) + (1 - p_G)U(y; B)$$

For any beliefs $(p_G, 1 - p_G)$ the investor may have, they will choose y optimally which means that they will choose y to solve the first order condition,

$$p_G U'(y; G) + (1 - p_G)U'(y; B) = 0$$

which means $y(p_G)$ is the expected utility maximizing choice and the maximal utility is,

$$V(p_G) = p_G U(y(p_G); G) + (1 - p_G)U(y(p_G); B)$$

Now suppose that the probability of G and B depend upon a previous choice, x, made by the investor (that is fixed when they choose y) and the outcome of a research effort that is a function x with outcomes H and L. Given a choice of x the probability of research outcome H is $\rho_H(x)$ and the probability of L is $1 - \rho_H(x)$. When the investor's research effort yields an H the probability of G is $p_G^H(x)$. When the investor's research effort yields an L the probability of G is $p_G^L(x)$. The pre-investment-decision value of choosing a value x is,

$$F(x) = \rho(x)V(p_G^H(x)) + (1 - \rho_H(x))V(p_G^L(x)) - C(x)$$

Where $C(x)$ is the cost from choosing level x.

Differentiate the composite value function above with respect to x and simplify the expression. (Hint: remember the first order condition defined above. This might be more complicated than you think.)

Problem 3.

Use the definitions below to prove the required proposition.

Definition 1. A preference relation $\succsim$ on a choice set X is *locally nonsatiated* if for every $\mathbf{x} \in X$ and every $\epsilon > 0$, there is a $\mathbf{y} \in X$ such that $||\mathbf{y} - \mathbf{x}|| \leq \epsilon$ and $\mathbf{y} \succ \mathbf{x}$.

Definition 2. A consumer has rational preference relation $\succsim$ on X represented by utility function $u(\mathbf{x})$. The prices for the goods in $\mathbf{x}$ are $\mathbf{p}$ and the consumer has wealth $w > 0$. The consumer's utility maximization problem is

$$\max_{\mathbf{x}} u(\mathbf{x}) \quad s.t. \quad \mathbf{p} \cdot \mathbf{x} \leq w$$

The solution (demand) is $x(\mathbf{p}, w)$ defined as the set of all $x \in X$ such that x maximizes utility subject to the budget constraint.

$$x(\mathbf{p}, w) = \{\mathbf{x} \in X : x \text{ is optimal and } \mathbf{p} \cdot \mathbf{x} \leq w\}$$

Proposition 1 (Walras' Law)**.** *If a consumer's preferences are locally nonsatiated and* $\mathbf{x}^* \in x(\mathbf{p}, w)$*, then* $\mathbf{p} \cdot \mathbf{x}^* = w$*.*

(Intuitively, if a consumption bundle is optimal, then it will involve the consumer spending all their income.) Hint: Try proof by contradiction.

Problem 4.

An economic agent is pricing the value of a new business they wish to purchase that generates annual returns. The present discounted value of the business changes over time, and the agent knows that the price they will be charged to purchase it will also change with time. In particular, the buyer knows the fundamental pricing equation,

$$p(t) = \int_t^\infty e^{(r+d)t} v(s) e^{-(r+d)s} \, \mathrm{d}s$$

Differentiate the price $p(t)$ **with respect to** t **and then solve for** $v(t)$ **in terms of** $p(t)$ **and** $p'(t)$**.**