Games, Strategies and Payoffs

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1. a list of players,

2. a **complete** description of what the players can do (their possible actions),

3. a description of what the players know **when** they act,

4. a specification of how the players’ actions lead to outcomes, and

5. a specification of the players’ **preferences over outcomes.**
1. Set of players: \( \{1, 2, \ldots, N\} \) (if finite),
2. Set of actions \( A_i \) for \( i = 1, \ldots, N \),
3. Collection of information sets \( \mathcal{I}_i = \{I_\alpha\} \) for \( i = 1, \ldots, N \),
4. For action profiles, \( a \in A = \times_{i=1}^{N} A_i \), and a set of outcomes \( O \) the function \( g : A \to O \).
5. For each player a preference relation \( \succeq_i \) over \( O \) represented by utility function \( u_i \) such that for \( x, y \in O \) we have \( x \succeq_i y \) if and only if \( u_i(x) \geq u_i(y) \).
Two Basic Game Forms

Extensive Form

- Decision nodes and edges (directed graph)
- Models simultaneous, dynamic, sequential and continuous games
- Transformable into strategic form
- Contains the “most” information

Strategic Form

- Matrix representation
- Typically simultaneous
- Players choose strategy “once and for all”
Decision nodes - opportunity to act, or time to make decision

Initial node (root) - **only one**

Branches (edges) - represent actions available at decision node

Terminal node - a unique end to the game

Each unique path through tree is an outcome
Discrete vs Continuous Action

![Diagram illustrating discrete vs continuous action](image-url)
**Tree Rule 1**: Every node is a successor of the initial node, and the initial node is the only one with this property.

- A **path** is a sequence of nodes that
  1. starts with initial node
  2. ends with a terminal node
  3. successive nodes in the sequence are immediate successors of each other.

- Each path has a unique terminal node (paths don’t cross)

**Tree Rule 2**: Each node except the initial node has exactly one immediate predecessor. The initial node has no predecessors.
Tree Rule 3: Multiple branches extending from the same node have different action labels.

Tree Rule 4: Each information set contains decision nodes for only one of the players.

Tree Rule 5: All nodes in a given information set must have the same number of immediate successors and they must have the same set of action labels on the branches leading to these successors.
Valid Game Trees
Rational Choice (Ordinal)

- A set $A$ of **actions** from which the decision-maker makes a choice.
- A set $C$ of possible **consequences** of these actions.
- A **consequence function** $g : A \rightarrow C$ that associates a consequence with each action.
- A **preference relation** $\succeq$ on the set $C$.

Ordinal utility means only **order** (greater number) matters, not “how much”
Expected Utility (Cardinal)

With cardinal utility “how much better” matters (up to affine transformation).

- Beliefs are represented by probabilities.
- We make assumptions on preferences to obtain expected utility form
  - \( x, y \in O \) and two beliefs over \( x, y \): \((p, 1 - p), (q, 1 - q)\).
  - If \( x \succ_i y \), then what about
    \[ u(x)p + u(y)(1 - p) \succeq ? \preceq u(x)q + u(y)(1 - q) \]
  - \((p, 1 - p) \succeq (q, 1 - q)\) if and only if
    \[ u(x)p + u(y)(1 - p) \geq u(x)q + u(y)(1 - q) \]
- Only consider beliefs that are consistent with rationality
  - Experiential Learning
  - Simulated Introspection
Information Sets

- In any game $G$, each player has a list of decision nodes they can act (move) at.
- A partition of this list is a set of groupings that determine which nodes the player can distinguish between.
- Two nodes in the same set cannot be distinguished from each other.
- Each group is an information set.

Perfect vs Imperfect Information

- If every information set, of every player, is a singleton, then its perfect information.
- **Critical Rule**: The actions at each node in the same information set must be identical.
Subgames

Our description of the extensive form allows us to model Games-within-Games.

- **Subtree** - a nonterminal node together with all ensuing nodes

- **Regular subtree** - a type of subtree that contains all information sets that have at least one node in the subtree.

- A **subgame** is a regular subtree with associated payoffs.

- A **substrategy** for a subgame is that part of a strategy which prescribes behavior only for information sets in that subgame.
Critical Rule of Finding Subgames

Given an extensive-form game \( \Gamma \), a decision node \( x \) in the tree is said to \textit{initiate} a \textbf{subgame} if neither \( x \) nor any of its successors are in an \textbf{information set that contains nodes that are not successors} of \( x \). A \textbf{subgame} is the tree structure defined by such a node \( x \) and its successors.
Valid Subgames

ISA

Don't Kidnap

Kidnap

Don't Pay

Pay

ISA

 ISA

 ISA

 Kill

 Release

 Kill

 Release

 (a, b)

 (c, d)

 (e, f)

 (g, h)

 (i, j)
Valid Subgames

- **ISIS**
  - Don’t Kidnap
    - $(a, b)$
  - Kidnap
    - You
      - Don’t Pay
        - $(c, d)$
      - Pay
        - Kill
          - $(e, f)$
        - Release
          - Kill
            - $(g, h)$
          - Release
            - $(i, j)$
A strategy for player $i$ is a **complete contingent** plan for how to play the game.

- **Complete**: specifies an action to take **at every information set** of player $i$.
- **Contingent**: specifies behavior even at **information sets that are never reached**.

The set $S_i$ is the set of all possible strategies player $i$ can choose in game $G$ and is called a **strategy set**.

A list $(s_1, s_2, \ldots, s_N)$ where $s_i \in S_i$ for all $i = 1, \ldots, N$ is a **strategy profile**.
Strategic Form Game

1. A list of players \( \{1, 2, \ldots, N\} \)
2. A strategy set \( S_i \) for every player
3. Each strategy profile specifies an outcome of the game
4. Each player has preferences (payoffs) over outcomes.

- With two players, the game can be represented with a single matrix
- Player 1’s strategies are listed in the rows, player 2’s strategies are listed in the columns, the cells specify the payoffs.
• History of play since game started.
• Perfect vs imperfect recall of game history.
• **Key Assumption**: The game is initially understood and agreed on by the players; the rules of the game are **common knowledge**.
• Once in the game, players are *not* allowed to change the game.
• However, we (the modeler) can enrich the game and give players more options.
Population of drivers, $N = 4000$, $A$ drivers take START-A-END route and $B$ drivers take START-B-END route. Drivers want to minimize driving time.
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- START-A-END: $A/100 + 45$
- START-B-END: $B/100 + 45$
Population of drivers $N = 4000$, $A$ drivers take START-A-END route and $B$ drivers take START-B-END route.

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Drivers want to minimize driving time.
Braess’ Paradox

\[ \frac{A}{100} + 45 = \frac{B}{10} + 45 \] implies that \( A = B \). Since \( A + B = 4000 \), we know \( A = 2000 \) and \( B = 2000 \). Stable (equilibrium) commute time is 65 minutes.
Clever player “discovers” short-cut and now her commute is approximately 40 minutes! Everyone else still drives for about 65 minutes, what a bunch of morons!
Once everyone “gets wise” they all want to take the short-cut. Equilibrium travel time is $\frac{4000}{100} + \frac{4000}{100} = 40 + 40 = 80$ minutes! All other routes take 85 minutes so no one has incentive to unilaterally change. This result is a **stable outcome**. Tricky advantages can be fleeting.