

From Bayesian Nash Equilibrium (BNE) to Perfect Bayesian Equilibrium (PBE)

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BNEs and Sequential rationality

- So far we have learned how to find BNEs in incomplete information games.
 - We are doing great!
- In settings where players are uncertain about their opponent's types. . .
 - this is a fantastic solution concept.
 - since it specifies optimal strategies, given the information every player has access to.

BNEs and Sequential rationality

- What if player interact in a sequential-move game?
 - Can the BNE prescribe "insensible" behavior? Yes!
 - But, what do we mean by "insensible" behavior?
 - Strategies that are not sequentially rational.
 - We will, hence, need a solution concept that guarantees sequential rationality (as SPNE, but applied to contexts of incomplete information).
- Let's show this with an example.
- **Use now the separate handout:**
 - "Why do we need Perfect Bayesian Equilibrium? Asking for sequential rationality in sequential-move games with incomplete information."

More examples about how to find PBEs

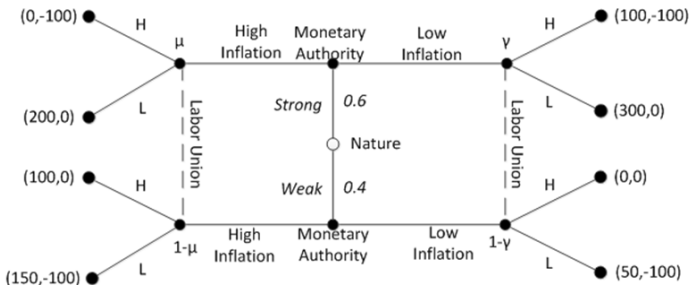
- After finding the PBEs in the Gift game...
- Let's now practice with another example (*Monetary Authority game*):
 - Now we will consider a Strong or Weak monetary authority, who makes an inflation announcement.
 - And a labor union (uninformed about the monetary authority's true commitment with low inflation policies, either Strong or Weak)...
 - decides whether to demand large or moderate salary increases.

Monetary authority game

- **Example:** Let us consider the following sequential game with incomplete information:
 - A monetary authority (such as the Federal Reserve Bank) privately observes its real degree of commitment with maintaining low inflation levels.
 - After knowing its type (either Strong or Weak), the monetary authority decides whether to announce that the expectation for inflation is either High or Low.
 - A labor union, observing the message sent by the monetary authority, decides whether to ask for high or low salary raises (denoted as H or L, respectively)

Monetary authority game

- The game tree that represents this incomplete information game is, hence, as follows:



PBEs-Monetary Authority

- Before starting to find all possible PBEs...
 - Let us briefly set up our "road map"
- That is, let's recall the 5-step procedure that we need to follow in order to find PBEs.

Procedure to find PBEs

1. Specify a strategy profile for the privately informed player, either separating or pooling.
 - In our above example, there are only four possible strategy profiles for the privately informed monetary authority: two separating strategy profiles, $High^S Low^W$ and $Low^S High^W$, and two pooling strategy profiles, $High^S High^W$ and $Low^S Low^W$.
 - (For future reference, it might be helpful to shade the branches corresponding to the strategy profile we test.)
2. Update the uninformed player's beliefs using Bayes' rule, whenever possible.
 - In our above example, we need to specify beliefs μ and γ , which arise after the labor union observes a high or a low inflation announcement, respectively.

Procedure to find PBEs - Cont'd

3. Given the uninformed player's updated beliefs, find his optimal response.
 - In our above example, we first determine the optimal response of the labor union (H or L) upon observing a high-inflation announcement (given its updated belief μ),
 - we then determine its optimal response (H or L) after observing a low-inflation announcement (given its updated belief γ).
 - (Also for future reference, it might be helpful to shade the branches corresponding to the optimal responses we just found.)

Procedure to find PBEs - Cont'd

4. Given the optimal response of the uninformed player, find the optimal action (message) for the informed player.
 - In our previous example, we first check if the Strong monetary authority prefers to make a high or low inflation announcement (given the labor union's responses determined in step 3).
 - We then operate similarly for the Weak type of monetary authority.

Procedure to find PBEs - Cont'd

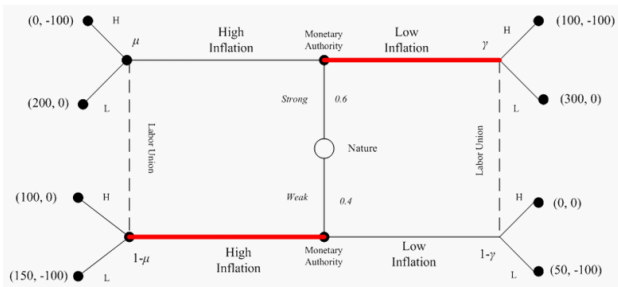
5. Then check if this strategy profile for the informed player coincides with the profile suggested in step 1.
 - If it coincides, then this strategy profile, updated beliefs and optimal responses **can** be supported as a PBE of the incomplete information game.
 - Otherwise, we say that this strategy profile **cannot** be sustained as a PBE of the game.

Procedure to find PBEs - Cont'd

- Let us next separately apply this procedure to test each of the four candidate strategy profiles:
 - two separating strategy profiles:
 - $High^S Low^W$, and
 - $Low^S High^W$.
 - And two pooling strategy profiles:
 - $High^S High^W$, and
 - $Low^S Low^W$.

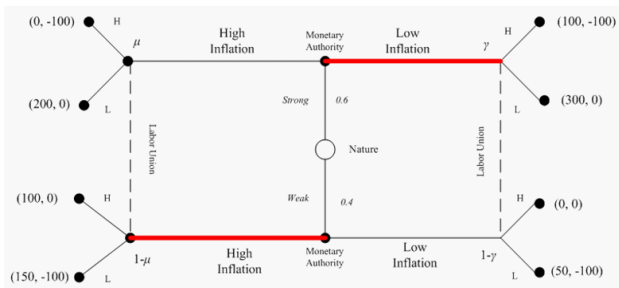
Separating equilibrium with (Low,High)

- Let us first check separating strategy profile: $Low^S High^W$.



- Step #1:** Specifying strategy profile $Low^S High^W$ that we will test.
 - (See shaded branches in the figure.)

Separating equilibrium with (Low,High)

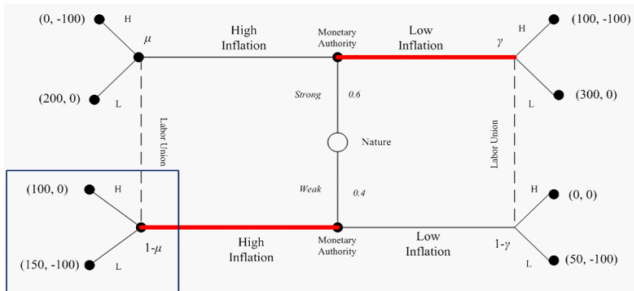


• Step #2: Updating beliefs

- (a) After high inflation announcement (left-hand side)

$$\mu = \frac{0.6\alpha^{Strong}}{0.6\alpha^{Strong} + 0.4\alpha^{Weak}} = \frac{0.6 \times 0}{0.6 \times 0 + 0.4 \times 1} = 0$$

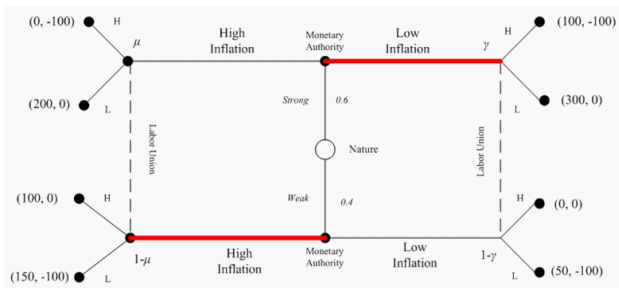
Separating equilibrium with (Low,High)



• Step #2: Updating beliefs

- This implies that after high inflation...
- the labor union restricts its belief to the lower left-hand corner (see box), since $\mu = 0$ and $1 - \mu = 1$

Separating equilibrium with (Low,High)

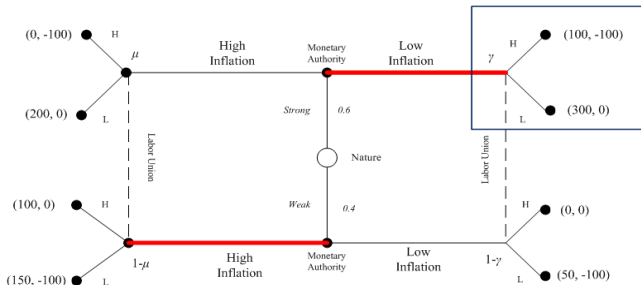


• Step #2: Updating beliefs

- (b) After low inflation announcement (right-hand side)

$$\gamma = \frac{0.6 (1 - \alpha^{Strong})}{0.6 (1 - \alpha^{Strong}) + 0.4 (1 - \alpha^{Weak})} = \frac{0.6 \times 1}{0.6 \times 1 + 0.4 \times 0} = 1$$

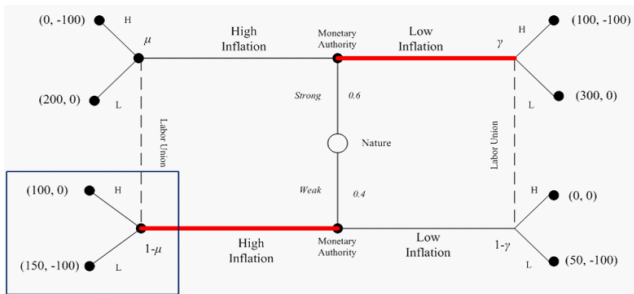
Separating equilibrium with (Low,High)



• Step #2: Updating beliefs

- This implies that, after low inflation...
- the labor union restricts its belief to the upper right-hand corner (see box), since $\gamma = 1$ and $1 - \gamma = 0$.

Separating equilibrium with (Low,High)

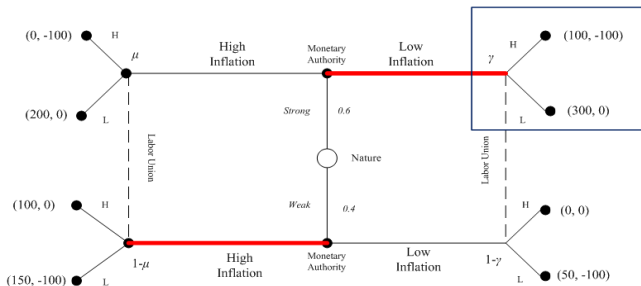


- **Step #3: Optimal response**
 - (a) After high inflation announcement, respond with H since

$$0 > -100$$

in the lower left-hand corner of the figure (see blue box).

Separating equilibrium with (Low,High)



- **Step #3: Optimal response**

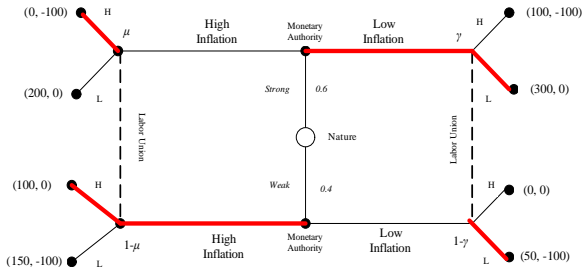
- (b) After low inflation announcement, respond with L since

$$0 > -100$$

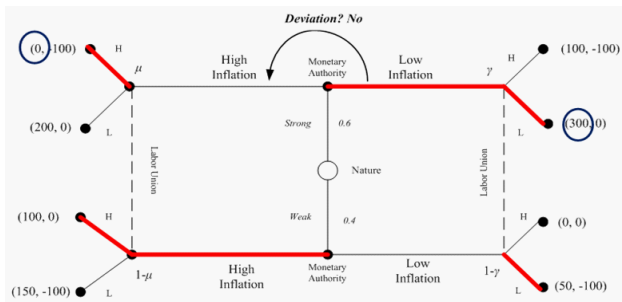
in the upper right-hand corner of the figure (see box).

Separating equilibrium with (Low,High)

- We can hence summarize the optimal responses we just found, by shading them in the figure:
 - H after high inflation, but L after low inflation.

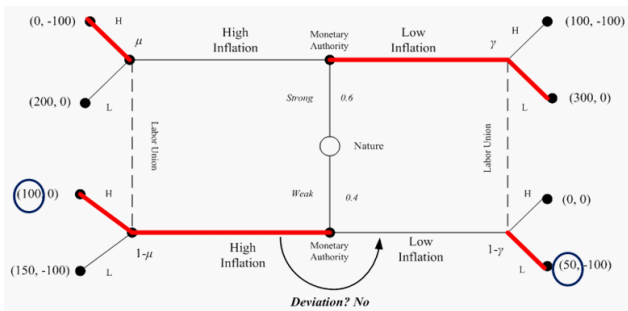


Separating equilibrium with (Low,High)



- **Step #4:** Optimal messages by the informed player
 - (a) When the monetary authority is Strong, if it chooses Low (as prescribed), its payoff is \$300,
 - while if it deviates, its payoff decreases to \$0.
 - (No incentives to deviate).

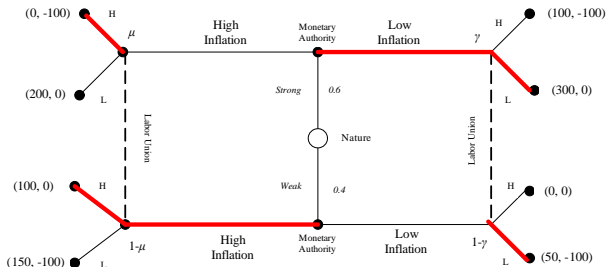
Separating equilibrium with (Low,High)



• Step #4: Optimal messages

- (b) When the monetary authority is Weak, if it chooses High (as prescribed), its payoff is \$100,
- while if it deviates, its payoff decreases to \$50.
- (No incentives to deviate either).

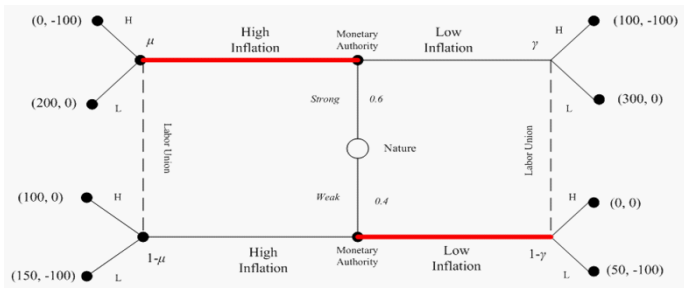
Separating equilibrium with (Low,High)



- Since no type of privately informed player (monetary authority) has incentives to deviate,
 - The separating strategy profile $Low^S High^W$ can be sustained as a PBE.

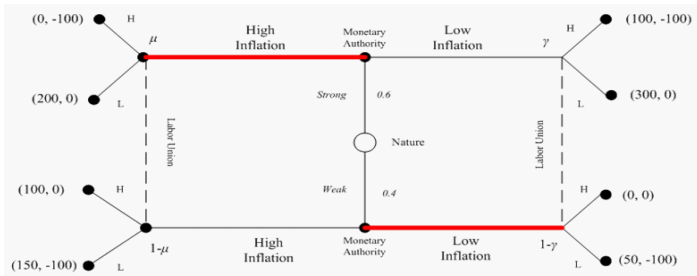
Separating equilibrium with (High,Low)

- Let us now check the opposite separating strategy profile: $High^S Low^W$.



- Step #1:** Specifying strategy profile $High^S Low^W$ that we will test.
 - (See shaded branches in the figure.)

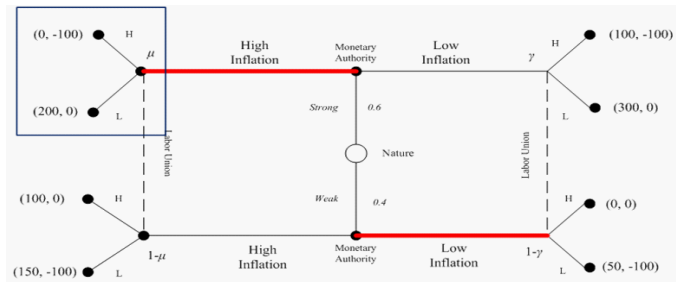
Separating equilibrium with (High,Low)



- **Step #2: Updating beliefs**
 - (a) After high inflation announcement

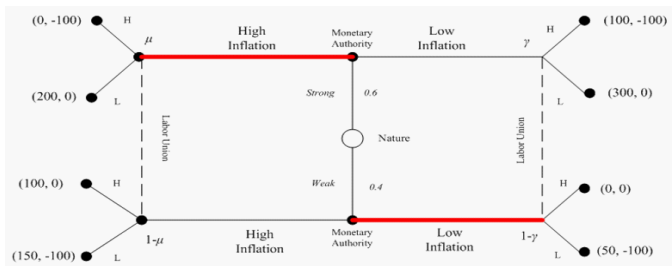
$$\mu = \frac{0.6\alpha^{Strong}}{0.6\alpha^{Strong} + 0.4\alpha^{Weak}} = \frac{0.6 \times 1}{0.6 \times 1 + 0.4 \times 0} = 1$$

Separating equilibrium with (High,Low)



- **Step #2: Updating beliefs**
 - Hence, after high inflation...
 - the labor union restricts its beliefs to $\mu = 1$ in the upper left-hand corner (see box).

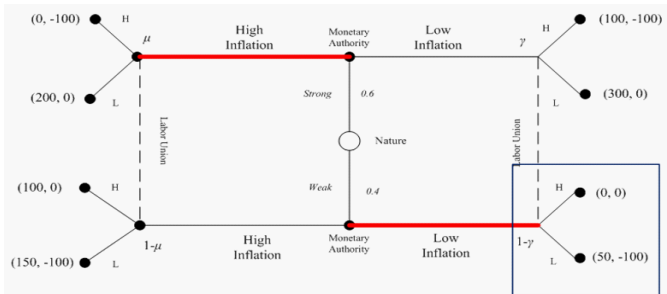
Separating equilibrium with (High,Low)



- **Step #2: Updating beliefs**
 - (b) After low inflation announcement

$$\gamma = \frac{0.6 (1 - \alpha^{Strong})}{0.6 (1 - \alpha^{Strong}) + 0.4 (1 - \alpha^{Weak})} = \frac{0.6 \times 0}{0.6 \times 0 + 0.4 \times 1} = 0$$

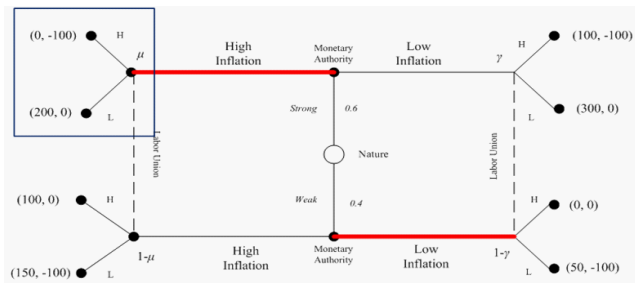
Separating equilibrium with (High,Low)



• Step #2: Updating beliefs

- Hence, after low inflation...
- the labor union restricts its beliefs to $\gamma = 0$ (i.e., $1 - \gamma = 1$) in the lower right-hand corner (see box).

Separating equilibrium with (High,Low)



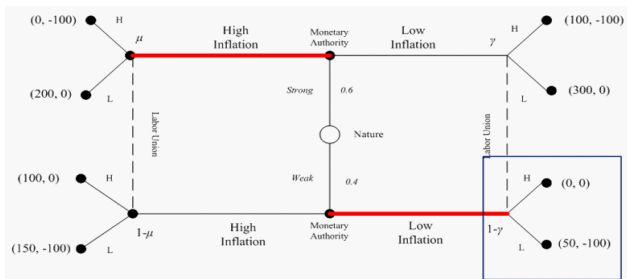
• Step #3: Optimal response

- (a) After high inflation announcement, respond with L since

$$0 > -100$$

in the upper left-hand corner of the figure (see box).

Separating equilibrium with (High,Low)



• Step #3: Optimal response

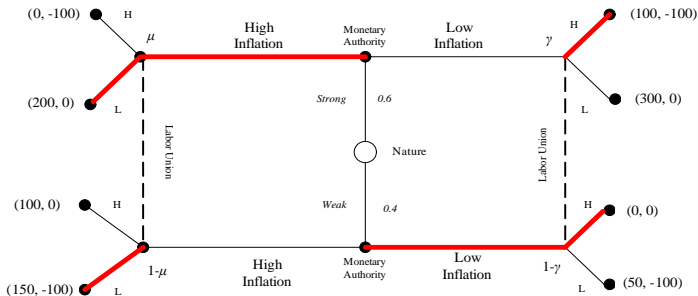
- (a) After low inflation announcement, respond with H since

$$0 > -100$$

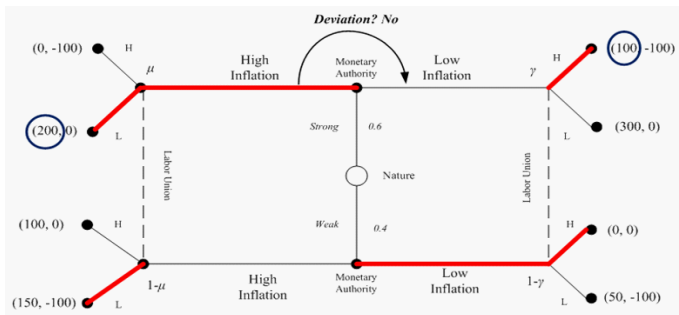
in the lower right-hand corner of the figure (see box).

Separating equilibrium with (High, Low)

- Summarizing the optimal responses we just found:
 - L after high inflation, but H after high inflation.

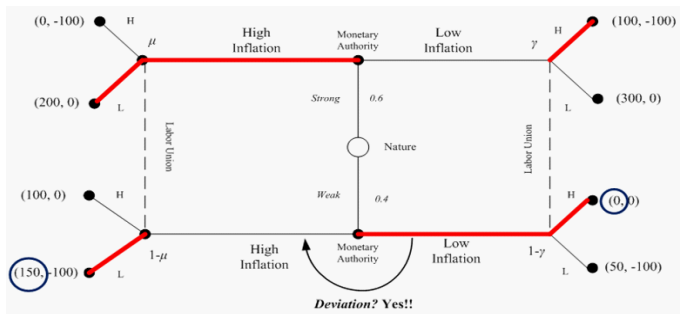


Separating equilibrium with (High,Low)



- **Step #4:** Optimal messages of the informed player
 - (a) When the monetary authority is Strong, if it chooses High (as prescribed), its payoff is \$200,
 - while if it deviates, its payoff decreases to \$100.
 - (No incentives to deviate).

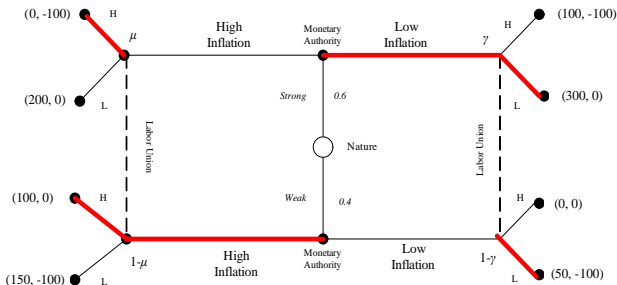
Separating equilibrium with (High,Low)



• Step #4: Optimal messages

- (b) When the monetary authority is Weak, if it chooses Low (as prescribed), its payoff is \$0,
- while if it deviates, its payoff **increases** to \$150.
- (Incentives to deviate!!).

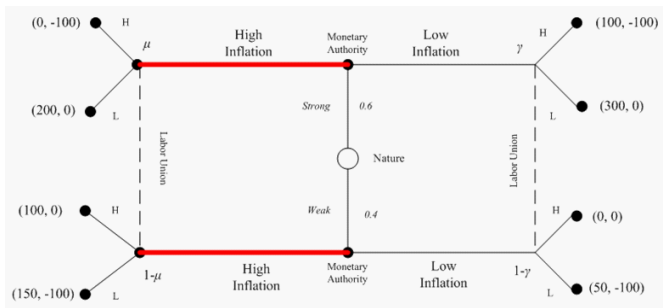
Separating equilibrium with (High,Low)



- Since we found one type of privately informed player (the Weak monetary authority) who has incentives to deviate...
 - The separating strategy profile *High*^S *Low*^W **cannot** be sustained as a PBE.

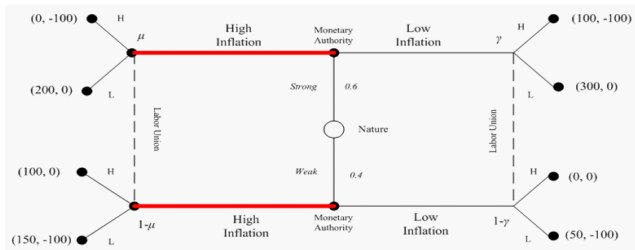
Pooling equilibrium with (High,High)

- Let us now test the pooling strategy profile $High^S High^W$.



- Step #1:** Specifying strategy profile $High^S High^W$ that we will test.
 - (See shaded branches in the figure.)

Pooling equilibrium with (High,High)

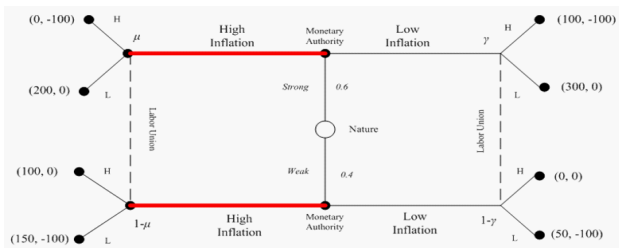


- **Step #2:** Updating beliefs
 - (a) After high inflation announcement

$$\mu = \frac{0.6\alpha^{Strong}}{0.6\alpha^{Strong} + 0.4\alpha^{Weak}} = \frac{0.6 \times 1}{0.6 \times 1 + 0.4 \times 1} = 0.6$$

so the high inflation announcement is uninformative.

Pooling equilibrium with (High,High)



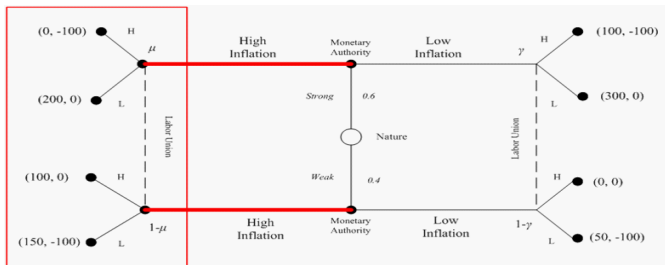
• Step #2: Updating beliefs

- (b) After low inflation announcement (off-the-equilibrium path)

$$\gamma = \frac{0.6 (1 - \alpha^{Strong})}{0.6 (1 - \alpha^{Strong}) + 0.4 (1 - \alpha^{Weak})} = \frac{0.6 \times 0}{0.6 \times 0 + 0.4 \times 0} = \frac{0}{0}$$

hence, $\gamma \in [0, 1]$.

Pooling equilibrium with (High,High)



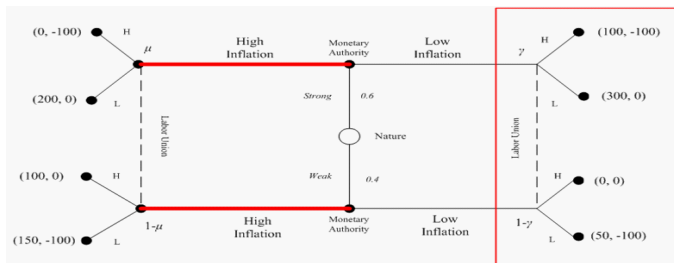
• Step #3: Optimal response

- (a) After high inflation announcement (along the equil. path), respond with L since

$$EU_{Labor}(H|High) = 0.6 \times (-100) + 0.4 \times 0 = -60$$

$$EU_{Labor}(L|High) = 0.6 \times 0 + 0.4 \times (-100) = -40$$

Pooling equilibrium with (High,High)



• Step #3: Optimal response

- (a) After low inflation announcement (off-the-equil.),

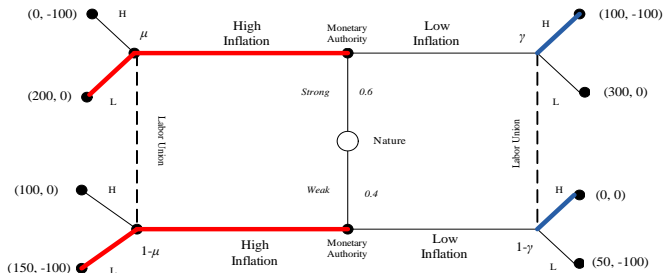
$$EU_{Labor}(H|Low) = \gamma \times (-100) + (1 - \gamma) \times 0 = -100\gamma$$

$$EU_{Labor}(L|Low) = \gamma \times 0 + (1 - \gamma) \times (-100) = -100 + 100\gamma$$

i.e., respond with H if $\gamma < \frac{1}{2}$.

Pooling equilibrium with (High,High)

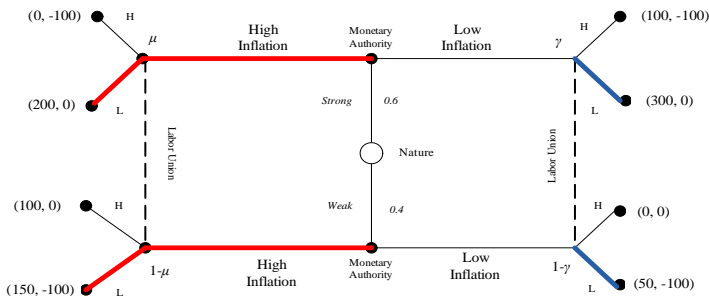
- Summarizing the optimal responses we found...
 - Note that we need to divide our analysis into two cases:
 - Case 1**, where $\gamma < \frac{1}{2}$, implying that the labor union responds with H after observing low inflation (right-hand side).



Pooling equilibrium with (High,High)

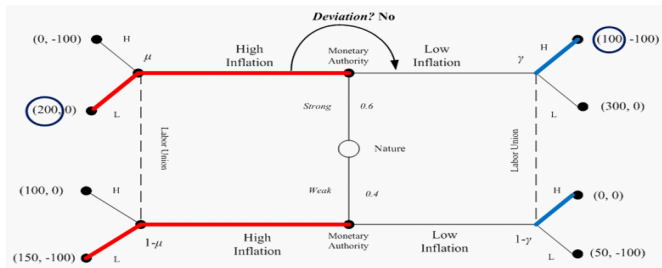
• and...

- **Case 2**, where $\gamma \geq \frac{1}{2}$, implying that the labor union responds with L after observing low inflation (right-hand side).



Pooling equilibrium with (High,High)

Case 1, where $\gamma < \frac{1}{2}$

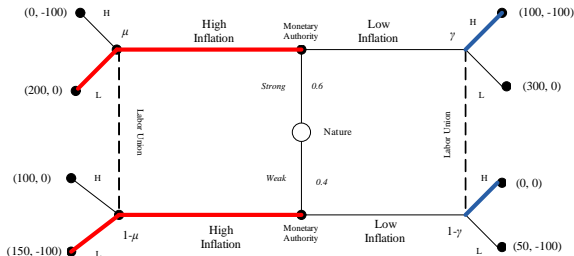


• Step #4: Optimal messages

- (a) When the monetary authority is Strong, if it chooses High (as prescribed), its payoff is \$200,
- while if it deviates, its payoff decreases to \$100.
- (No incentives to deviate).

Pooling equilibrium with (High,High)

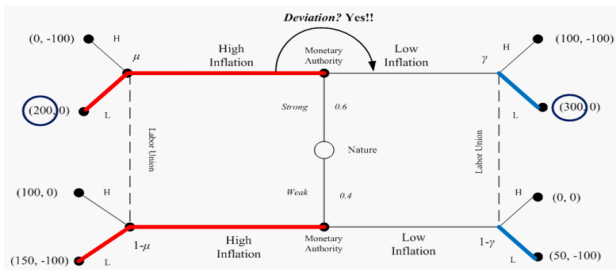
Case 1, where $\gamma < \frac{1}{2}$



- No type of monetary authority has incentives to deviate.
- Hence, the pooling strategy profile $High^S High^W$ **can** be sustained as a PBE when off-the-equilibrium beliefs satisfy $\gamma < \frac{1}{2}$.

Pooling equilibrium with (High,High)

Case 2, where $\gamma \geq \frac{1}{2}$

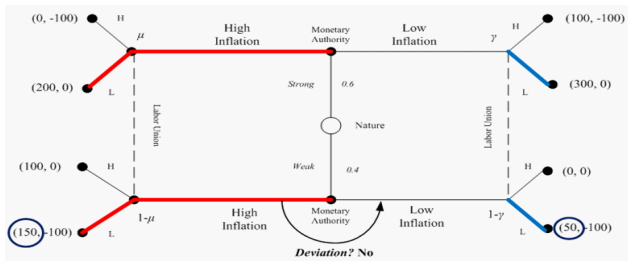


• Step #4: Optimal messages

- (a) When the monetary authority is Strong, if it chooses High (as prescribed), its payoff is \$200,
- while if it deviates, its payoff **increases** to \$300.
- (Incentives to deviate!!).

Pooling equilibrium with (High,High)

Case 2, where $\gamma \geq \frac{1}{2}$

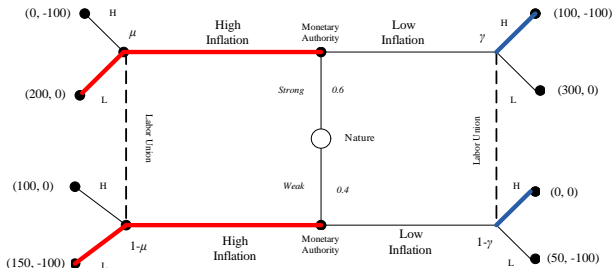


• Step #4: Optimal messages

- (b) When the monetary authority is Weak, if it chooses High (as prescribed), its payoff is \$150,
- while if it deviates, its payoff drops to \$50.
- (No incentives to deviate).

Pooling equilibrium with (High,High)

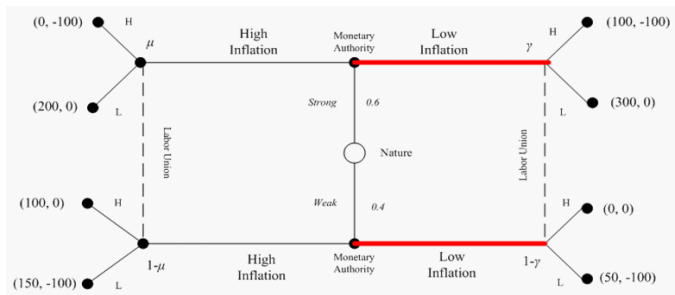
Case 2, where $\gamma \geq \frac{1}{2}$



- Since we found one type of privately informed player (the Strong monetary authority) who has incentives to deviate...
 - The pooling strategy profile $High^S High^W$ **cannot** be sustained as a PBE when off-the-equilibrium beliefs satisfy $\gamma \geq \frac{1}{2}$.

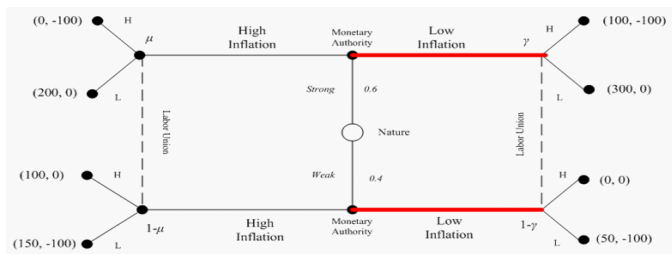
Pooling equilibrium with (Low,Low)

- Let us now examine the opposite pooling strategy profile.



- Step #1:** Specifying strategy profile $Low^S Low^W$ that we will test.
 - (See shaded branches in the figure.)

Pooling equilibrium with (Low,Low)

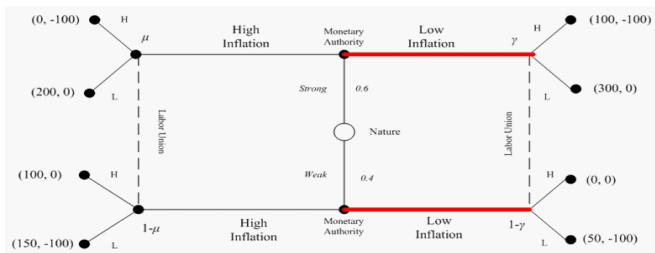


- **Step #2: Updating beliefs**
 - (a) After a low inflation announcement

$$\gamma = \frac{0.6 (1 - \alpha^{Strong})}{0.6 (1 - \alpha^{Strong}) + 0.4 (1 - \alpha^{Weak})} = \frac{0.6 \times 1}{0.6 \times 1 + 0.4 \times 1} = 0.6$$

so posterior and prior beliefs coincide.

Pooling equilibrium with (Low,Low)



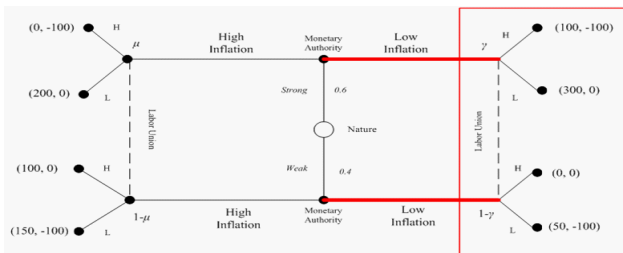
• Step #2: Updating beliefs

- (b) After a high inflation announcement (off-the-equil. path)

$$\mu = \frac{0.6\alpha^{Strong}}{0.6\alpha^{Strong} + 0.4\alpha^{Weak}} = \frac{0.6 \times 0}{0.6 \times 0 + 0.4 \times 0} = \frac{0}{0}$$

hence, $\mu \in [0, 1]$.

Pooling equilibrium with (Low,Low)



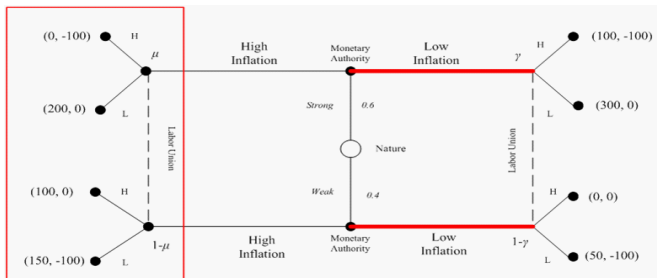
• Step #3: Optimal response

- (a) After a low inflation announcement (along the equilibrium path), respond with L since

$$EU_{Labor}(H|Low) = 0.6 \times (-100) + 0.4 \times 0 = -60$$

$$EU_{Labor}(L|Low) = 0.6 \times 0 + 0.4 \times (-100) = -40$$

Pooling equilibrium with (Low,Low)



• Step #3: Optimal response

- (a) After a high inflation announcement (off-the-equil.),

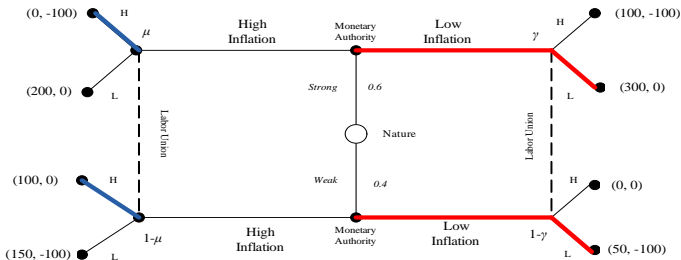
$$EU_{Labor}(H|Low) = \mu \times (-100) + (1 - \mu) \times 0 = -100\mu$$

$$EU_{Labor}(L|Low) = \mu \times 0 + (1 - \mu) \times (-100) = -100 + 100\mu$$

i.e., respond with H if $\mu < \frac{1}{2}$.

Pooling equilibrium with (Low,Low)

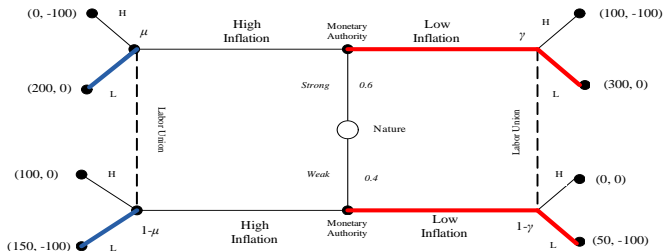
- Summarizing the optimal responses we found...
 - Note that we need to divide our analysis into two cases:
 - Case 1**, where $\mu < \frac{1}{2}$, implying that the labor union responds with H after observing high inflation (left-hand side).



Pooling equilibrium with (Low,Low)

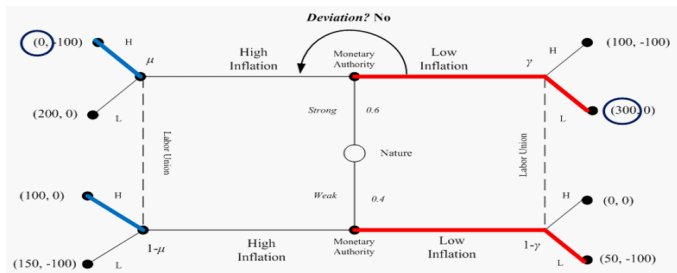
• and...

- **Case 2**, where $\mu \geq \frac{1}{2}$, implying that the labor union responds with L after observing high inflation (left-hand side).



Pooling equilibrium with (Low,Low)

Case 1, where $\mu < \frac{1}{2}$

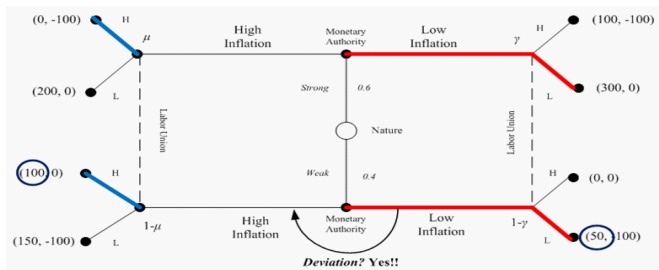


Step #4: Optimal messages

- (a) When the monetary authority is Strong, if it chooses Low (as prescribed), its payoff is \$300,
- while if it deviates, its payoff decreases to \$200.
- (No incentives to deviate).

Pooling equilibrium with (Low,Low)

Case 1, where $\mu < \frac{1}{2}$

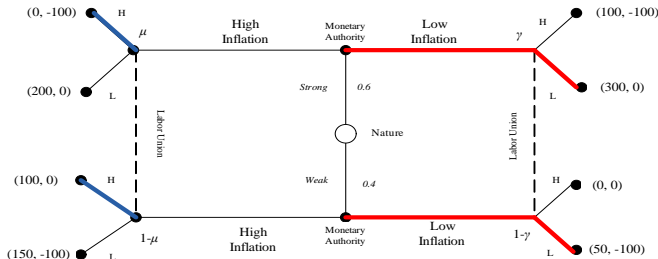


Step #4: Optimal messages

- (b) When the monetary authority is Weak, if it chooses High (as prescribed), its payoff is \$50,
- while if it deviates, its payoff **increases** to \$100.
- (Incentives to deviate!!).

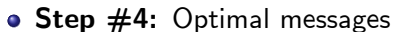
Pooling equilibrium with (Low,Low)

Case 1, where $\mu < \frac{1}{2}$



- Since we found one type of privately informed player (the Weak monetary authority) who has incentives to deviate...
 - The pooling strategy profile $Low^S Low^W$ **cannot** be sustained as a PBE when off-the-equilibrium beliefs satisfy $\mu < \frac{1}{2}$

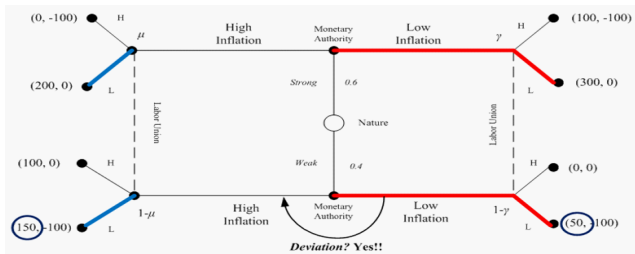
Case 2, where $\mu \geq \frac{1}{2}$



- A set of small navigation icons typically found in Beamer presentations, including symbols for back, forward, search, and other slide controls.

Pooling equilibrium with (Low,Low)

Case 2, where $\mu \geq \frac{1}{2}$

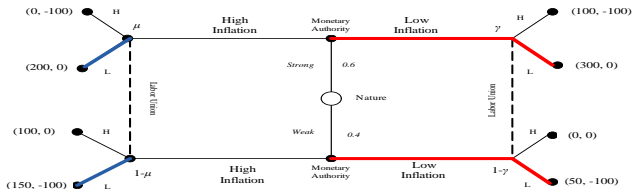


• Step #4: Optimal messages

- (b) When the monetary authority is Weak, if it chooses Low (as prescribed), its payoff is \$50,
- while if it deviates, its payoff **increases** to \$150.
- (Incentives to deviate!!).

Pooling equilibrium with (Low,Low)

Case 2, where $\mu \geq \frac{1}{2}$



- Since we found one type of privately informed player (the Weak monetary authority) who has incentives to deviate...
 - The pooling strategy profile $Low^S Low^W$ **cannot** be sustained as a PBE when off-the-equilibrium beliefs satisfy $\mu \geq \frac{1}{2}$.
- Hence, the pooling strategy profile $Low^S Low^W$ **cannot** be sustained as a PBE for **any** off-the-equilibrium beliefs μ .