Mixed strategy equilibria (msNE) with N players

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EconS 424 - Strategy and Game Theory
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We learned how to find msNE in games:

- with 2 players, each with 2 available strategies (2x2 matrix)
  - e.g., matching pennies game, battle of the sexes, etc.
- with 2 players, but each having 3 available strategies (3x3 matrix)
  - e.g., tennis game (which actually reduced to a 2x2 matrix after deleting strictly dominated strategies), and
  - the rock-paper-scissors game, where we couldn’t identify strictly dominated strategies and, hence, had to make players indifferent between their three available strategies.

What about games with 3 players?
More advanced mixed strategy games

What if we have three players, instead of two? (Harrington pp 201-204). "Friday the 13th!"
More advanced mixed strategy games

<table>
<thead>
<tr>
<th></th>
<th>Front</th>
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<tbody>
<tr>
<td><strong>Tommy</strong></td>
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<tr>
<td>Front</td>
<td>0, 0, 0</td>
<td>-4, 1, 2</td>
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<td>1, -4, 2</td>
<td>2, 2, -2</td>
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**Beth**

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<tr>
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<tbody>
<tr>
<td><strong>Jason, Front</strong></td>
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<tr>
<td>Front</td>
<td>3, 3, -2</td>
<td>1, -4, 2</td>
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More advanced mixed strategy games

Friday the 13th!

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<td>0, 0, 0</td>
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First step: let’s check for strictly dominated strategies (none).

Second step: let’s check for psNE (none). The movie is getting interesting!

Third step: let’s check for msNE. (note that all strategies are used by all players), since there are no strictly dominated strategies.
msNE with three players

Since we could not delete any strictly dominated strategy, then all strategies must be used by all three players.

In this exercise we need three probabilities, one for each player.

Let’s denote:

- $t$ the probability that Tommy goes through the front door (first row in both matrices).
- $b$ the probability that Beth goes through the front door (first column in both matrices).
- $j$ the probability that Jason goes through the front door (left-hand matrix).
Let us start with Jason, $EU_J(F) = EU_J(B)$, where

$$EU_J(F) = \underbrace{tb_0 + t(1-b)2} + \underbrace{(1-t)b2 + (1-t)(1-b)(-2)}$$

Tommy goes through the front door, $t$

$= -2 + 4t + 4b - 6tb$

and

$$EU_J(B) = tb(-2) + t(1-b)2 + (1-t)b2 + (1-t)(1-b)0$$

$= 2t + 2b - 6tb$

since $EU_J(F) = EU_J(B)$ we have

$$-2 + 4t + 4b - 6tb = 2t + 2b - 6tb \iff t + b = 1 \quad (1)$$
msNE with three players

Let us now continue with Tommy, $EU_T(F) = EU_T(B)$, where

$$EU_T(F) = bj0 + (1 - b)j(-4) + b(1 - j)3 + (1 - b)(1 - j)(1)$$
$$= 1 + 2b - 5j + 2bj$$

and

$$EU_T(B) = bj1 + (1 - b)j2 + b(1 - j)(-4) + (1 - b)(1 - j)(0)$$
$$= -4b + 2j + 3bj$$

since $EU_T(F) = EU_T(B)$ we have

$$1 + 2b - 5j + 2bj = -4b + 2j + 3bj \iff 7j - 6b + bj = 1 \quad (2)$$
And given that the payoffs for Tommy and Beth are symmetric, we must have that Tommy and Beth’s probabilities coincide, $t = b$.

Hence we don’t need to find the indifference condition $EU_B(F) = EU_B(B)$ for Beth.

Instead, we can use Tommy’s condition (2) (i.e., $7j - 6b + bj = 1$), to obtain the following condition for Beth:

$$7j - 6t + tj = 1$$

We must solve conditions (1),(2) and (3).
First, by symmetry we must have that $t = b$. Using this result in condition (1) we obtain

\[ t + b = 1 \implies t + t = 1 \implies t = b = \frac{1}{2} \]

Using this result into condition (2), we find

\[ 7j - 6b + bj = 7j - 6\frac{1}{2} + \frac{1}{2}j = 1 \]

Solving for $j$ we obtain $j = \frac{8}{15}$. 
msNE with three players

- Representing the msNE in Friday the 13th:

\[
\left\{ \left( \frac{1}{2} \text{Front}, \frac{1}{2} \text{Back} \right), \left( \frac{1}{2} \text{Front}, \frac{1}{2} \text{Back} \right), \left( \frac{8}{15} \text{Front}, \frac{7}{15} \text{Back} \right) \right\}
\]

- Tommy
- Beth
- Jason
msNE with three players

- **Just for fun:** What is then the probability that Tommy and Beth escape from Jason?
  - They escape if they both go through a door where Jason is not located.

\[
\begin{align*}
\frac{1}{2} \times \frac{1}{2} & \times \frac{8}{15} + \frac{1}{2} \times \frac{1}{2} \times \frac{7}{15} = \frac{15}{60}
\end{align*}
\]

- Jason goes Front

- Jason goes Back

- The **first term** represents the probability that both Tommy and Beth go through the Back door (which occurs with \(\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}\) probability) while Jason goes to the Front door.

- The **second term** represents the opposite case: Tommy and Beth go through the Front door (which occurs with \(\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}\) probability) while Jason goes to the Back door.
msNE with three players

- Even if they escape from Jason this time, there is still...

- There are actually NO sequels:
  - Their probability of escaping Jason is then \( \left( \frac{15}{60} \right)^{10} \), about 1 in a million!
A natural question at this point is how we can empirically test, as external observers, if individuals behave as predicted by our theoretical models.

In other words, how can we check if individuals randomize with approximately the same probability that we found to be optimal in the msNE of the game?
Testing the Theory

In order to test the theoretical predictions of our models, we need to find settings where players seek to "surprise" their opponents (so playing a pure strategy is not rational), and where stakes are high.

- Can you think of any?
Penalty kicks in soccer
Penalty kicks in soccer

His payoffs represent the probability that the kicker does not score (That is why within a given cell, payoffs sum up to one).

<table>
<thead>
<tr>
<th></th>
<th>Goalkeeper</th>
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<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td><strong>Kicker</strong></td>
<td></td>
</tr>
<tr>
<td>Left</td>
<td>.65, .35</td>
</tr>
<tr>
<td>Center</td>
<td>.95, .05</td>
</tr>
<tr>
<td>Right</td>
<td>.95, .05</td>
</tr>
</tbody>
</table>

Payoffs represent the probability he scores.
Penalty kicks in soccer

- We should expect soccer players randomize their decision.
  - Otherwise, the kicker could anticipate where the goalie dives and kick to the other side. Similarly for the goalie.

- Let’s describe the kicker’s expected utility from kicking the ball left, center or right.
Penalty kicks in soccer

\[
EU_{\text{Kicker}}(\text{Left}) = g_l \times 0.65 + g_r \times 0.95 + (1 - g_r - g_l) \times 0.95 \\
= 0.95 - 0.3g_l
\]  

(1)

\[
EU_{\text{Kicker}}(\text{Center}) = g_l \times 0.95 + g_r \times 0.95 + (1 - g_r - g_l) \times 0 \\
= 0.95(g_r + g_l)
\]  

(2)

\[
EU_{\text{Kicker}}(\text{Right}) = g_l \times 0.95 + g_r \times 0.65 + (1 - g_r - g_l) \times 0.95 \\
= 0.95 - 0.3g_r
\]  

(3)
Penalty kicks in soccer

Since the kicker must be indifferent between all his strategies,

\[ EU_{\text{Kicker}}(Left) = EU_{\text{Kicker}}(Right) \]

\[ 0.95 - 0.3g_l = 0.95 - 0.3g_r \implies g_l = g_r \implies g_l = g_r = g \]

Using this information in (2), we have

\[ 0.95(g + g) = 1.9g \]

Hence,

\[ \underbrace{0.95 - 0.3g}_{EU_{\text{Kicker}}(Left)} = \underbrace{1.9g}_{EU_{\text{Kicker}}(Center)} \implies g = \frac{0.95}{2.2} = 0.43 \]
Penalty kicks in soccer

Therefore,

\((\sigma_L, \sigma_C, \sigma_R) = (0.43, \underline{0.14}, \underline{0.43}) \)

From the fact that \(g_l + g_r + g_c = 1\), where \(g_l = g_r = g\)

If the set of goalkeepers is similar, we can find the same set of mixed strategies,

\((\sigma_L, \sigma_C, \sigma_R) = (0.43, 0.14, 0.43)\)
Penalty kicks in soccer

Hence, the probability that a goal is scored is:

- Goalkeeper dives left →
  \[0.43 \times (0.43 \times 0.65 + 0.14 \times 0.95 + 0.43 \times 0.95)\]

- Goalkeeper dives center →
  \[+0.14 \times (0.43 \times 0.95 + 0.14 \times 0 + 0.43 \times 0.95)\]

- Goalkeeper dives right →
  \[+0.43 \times (0.43 \times 0.95 + 0.14 \times 0.95 + 0.43 \times 0.65)\]

= 0.82044, i.e., a goal is scored with 82% probability.
Penalty kicks in soccer

- Interested in more details?
  - First, read Harrington pp. 199-201.
  - Then you can have a look at the article
  - This author published a very readable book last year:
Summarizing...

- So far we have learned how to find msNE in games:
  - with two players (either with 2 or more available strategies).
  - with three players (e.g., Friday the 13th movie).

- What about generalizing the notion of msNE to games with $N$ players?
  - Easy! We just need to guarantee that every player is indifferent between all his available strategies.
Example: "Extreme snob effect" (Watson).
Every player chooses between alternative X and Y (Levi’s and Calvin Klein). Every player i’s payoff is 1 if he selects Y, but if he selects X his payoff is:
- 2 if no other player chooses X, and
- 0 if some other player chooses X as well
Let's check for a symmetric msNE where all players select Y with probability $\alpha$. Given that player i must be indifferent between X and Y, $EU_i(X) = EU_i(Y)$, where

$$EU_i(X) = \frac{\alpha^{n-1}2}{\text{all other } n-1 \text{ players select } Y} + (1 - \alpha^{n-1})0 \quad \text{Not all other players select } Y$$
msNE with N players

and $EU_i(Y) = 1$, then $EU_i(X) = EU_i(Y)$ implies

$$\alpha^{n-1}2 = 1 \iff \alpha = \left(\frac{1}{2}\right)^{\frac{1}{n-1}}$$

Comparative statics of $\alpha$, the probability a player selects
the "conforming" option $Y$, $\alpha = \left(\frac{1}{2}\right)^{\frac{1}{n-1}}$:

- $\alpha$ increases in the size of the population $n$.

- That is, the larger the size of the population, the more likely it
is that somebody else chooses the same as you, and as a consequence you don’t take the risk of choosing the snob option $X$. Instead, you select the "conforming" option $Y$. 

msNE with N players

- Probability of choosing strategy Y as a function of the number of individuals, \( n \).

\[
\alpha = \left( \frac{1}{2} \right)^{n-1}
\]

\[
\text{prob}(X) + \text{prob}(Y) = 1, \quad \text{prob}(X) \quad \text{then,} \quad (X) = 1 - \text{prob}(Y)
\]
Another example of msNE with N players

- **Another example with N players: The bystander effect**
- The "bystander effect" refers to the lack of response to help someone nearby who is in need.
  - *Famous example*: In 1964 Kitty Genovese was attacked near her apartment building in New York City. Despite 38 people reported having heard her screams, no one came to her aid.
  - Also confirmed in laboratory and field studies in psychology.
Another example of msNE with N players

- General finding of these studies:
  - A person is less likely to offer assistance to someone in need when the person is in a large group than when he/she is alone.
    - e.g., all those people who heard Kitty Genovese’s cries knew that many others heard them as well.
    - In fact, some studies show that the more people that are there who could help, the less likely help is to occur.
  - Can this outcome be consistent with players maximizing their utility level?
    - Yes, let’s see how.
Another example of msNE with $N$ players

**Other players**

<table>
<thead>
<tr>
<th>Helps</th>
<th>All ignore</th>
<th>At least one helps</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Helps</td>
<td>a</td>
<td>c</td>
</tr>
<tr>
<td>Ignores</td>
<td>d</td>
<td>b</td>
</tr>
</tbody>
</table>

where $a > d$ → so if all ignore, I prefer to help the person in need.

but $b > c$ → so, if at least somebody helps, I prefer to ignore.

Note that assumptions are not so selfish: people would prefer to help if nobody else does.
Another example of msNE with N players

msNE:

Let’s consider a symmetric msNE whereby every player $i$:

- Helps with probability $p$, and
- Ignores with probability $1 - p$. 
Another example of msNE with N players

$$EU_i(\text{Help}) = \begin{cases} (1 - p)^{n-1} \ast a + [1 - (1 - p)^{n-1}] \ast c, \\ \quad \text{If everybody else ignores} \\ (1 - p)^{n-1} \ast d + [1 - (1 - p)^{n-1}] \ast b, \\ \quad \text{If at least one of the other } n-1 \text{ players helps} \end{cases}$$

$$EU_i(\text{Ignore}) = \begin{cases} (1 - p)^{n-1} \ast d + [1 - (1 - p)^{n-1}] \ast b, \\ \quad \text{If everybody else ignores} \\ (1 - p)^{n-1} \ast d + [1 - (1 - p)^{n-1}] \ast b, \\ \quad \text{If at least one of the other } n-1 \text{ players helps} \end{cases}$$

- When a player randomizes, he is indifferent between help and ignore,

$$EU_i(\text{Help}) = EU_i(\text{Ignore})$$

$$= (1 - p)^{n-1} \ast a + [1 - (1 - p)^{n-1}] \ast c$$

$$= (1 - p)^{n-1} \ast d + [1 - (1 - p)^{n-1}] \ast b$$

$$\Rightarrow (1 - p)^{n-1}(a - c - d + b) = b - c$$
Another example of msNE with N players

- Solving for $p$,

$$ (1 - p)^{n-1} = \frac{b - c}{a - c - d + b} $$

$$ \implies 1 - p = \left( \frac{b - c}{a - c - d + b} \right)^{\frac{1}{n-1}} $$

$$ \implies p^* = 1 - \left( \frac{b - c}{a - c - d + b} \right)^{\frac{1}{n-1}} $$

- Example: $a = 4$, $b = 3$, $c = 2$, $d = 1$, satisfying the initial assumptions: $a > d$ and $b > c$

$$ p^* = 1 - \left( \frac{3 - 1}{4 - 2 - 1 + 3} \right)^{\frac{1}{n-1}} = 1 - \left( \frac{1}{4} \right)^{\frac{1}{n-1}} $$
Another example of msNE with N players

- Probability of a person helping, $p^*$

More people makes me less likely to help.
Another example of msNE with $N$ players

- Probability that the person in need receives help, $(p^*)^n$

More people actually make it less likely that the victim is helped!
Intuitively, the new individual in the population brings a positive and a negative effect on the probability that the victim is finally helped:

- **Positive effect**: the additional individual, with his own probability of help, $p^*$, increases the chance that the victim is helped.

- **Negative effect**: the additional individual makes more likely, that someone will help the victim, thus leading each individual citizen to reduce his own probability of helping, i.e., $p^*$ decreases in $n$.

However, the fact that $(p^*)^n$ decreases in $n$ implies that the negative effect offsets the positive effect.