Exam 1
ECONS 424: Strategy and Game Theory

Spring 2017

General Directions: You have 50 minutes to complete the exam. You are allowed to write on the exam and use provided scratch paper. Please write and circle your answers on the exam clearly. Notes and phones are not allowed. You may use a simple or scientific calculator, but no graphing calculators. Please ask me for clarification on questions if you need any.

For questions 1,2 and 3, answer True or False. If the answer is False explain in a short answer why the statement is false.

Problem 1. (2 points) (True or False): In simultaneous games a rational player will not play a strictly dominated strategy, but in sequential games sometimes they will.

Solution: False: By definition, a strategy, call it \( s \), is strictly dominated by another strategy \( s' \) if \( u_i(s') > u_i(s) \) for any strategy choice of the other players. Because a player always does strictly better by playing \( s' \) than they would by playing \( s \), a rational player must always choose \( s' \) over \( s \). Hence, a rational player will never play a strictly dominated strategy in any type of game. Weakly dominated strategies are a different story.

Problem 2. (2 points) (True or False): A strategy \( s_i \) for player \( i \) in a game \( G \), will specify a single action for every one of player \( i \)'s information sets that are actually reached when \( s_{-i} \) is the strategy profile of the other players.

Solution: False: By definition, a strategy is a complete contingent plan for a player which describes exactly how they will act at ALL of their information sets, regardless of whether any player’s actions prevent certain information sets from being reached. This means it must specify one of the available actions at every single information set a player has.

Problem 3. (2 points) (True or False): In symmetric games network effects can lead to symmetric pure strategy Nash equilibria and congestion can lead to asymmetric pure strategy Nash equilibria.

Solution: True.
Problem 4. (2 points) (Circle the correct choice) Nash equilibrium requires that the rationality of the players be common knowledge so that players choose their strategies to maximize their payoffs given their beliefs about the strategies chosen by the other players AND it requires those beliefs to be
A.) rationalizable

B.) fair

C.) correct

D.) subjective

E.) none of the above

Problem 5. (2 points) (Circle the correct choice) Consider the game below played by Bob and Alice.

```
Bob
Formal Casual
Formal | 2, 3 | 1, 1
Casual | 0, 0 | 3, 2
```

The game facing Alice and Bob is a game of
A.) pure conflict

B.) constant sum conflict

C.) zero sum conflict

D.) coordination

E.) anti-coordination

F.) pursuit

G.) none of the above

Problem 6. (2 points) (Circle all that are correct) A strategic pre-commitment must have the properties:
A.) irreversible

B.) result in a higher security level for the player

C.) known and believed by other players

D.) established before other players make decisions

E.) A best response
**Problem 7.** (8 points) Study the extensive form game below and answer the following questions or problems in the space provided below. (Note in the game assume that for space reasons $W = War$ and $NW = No war$)

A.) Is this a game of perfect or imperfection information?

B.) How many players are in this game?

C.) Write down two pure strategies for each player in the game.

D.) How many pure strategies does each player in the game have?

E.) How many pure strategy profiles are there in the game?

F.) How many subgames are there in the game?

**Solution:**

A.) All information sets are not singletons, so this is a game of **imperfect information**.

B.) The game has 3 players: Iraq, UN, and US.

C.) Iraq has 3 information sets, each a singleton. Therefore each of Iraq’s strategies will specify exactly 3 actions, one for each information set. The UN has only a single information set with 2 actions so a strategy will specify just a single action (for the 1 information set). Finally, the US has a total of 4 information sets and so any strategy
specify a total of 4 actions. Below are two strategies from each player’s strategy set.

Iraq: $WMD/Deny/Deny, \quad WMD/Deny/Allow\&hide$

UN: $Inspect, \quad Do\ not\ inspect$

US: $War/War/War/No\ War, \quad No\ War/War/No\ War/No\ War$

D.) We can use the number of information sets for each player and the number of actions at each information set to determine the total number of strategies for each player. Iraq’s strategies are a list of 3 actions, one for each information set and there are 2 possible actions at two of the information sets and 3 available actions at the last. Hence, Iraq has $2 \times 2 \times 3 = 12$ possible strategies. The UN has only 2 possible strategies as it has only 1 information set with 2 actions. Finally, the US has 4 information sets, with available number of actions equal to 2 at each of those information sets. Hence, there are $2 \times 2 \times 2 \times 2 = 16$ pure strategies for the US. In summary

<table>
<thead>
<tr>
<th>Player</th>
<th>Strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iraq</td>
<td>12</td>
</tr>
<tr>
<td>UN</td>
<td>2</td>
</tr>
<tr>
<td>US</td>
<td>16</td>
</tr>
</tbody>
</table>

E.) Now that we know the total number of strategies for each player we can calculate the total number of possible pure strategy profiles for this game. Remember, a strategy profile is a list of strategies, exactly 1 for each player. In this case it will be a triple $(s_{Iraq}, s_{UN}, s_{US})$. The first component can take 12 different values, the second 2 possible values and finally the 3rd can take 16 different values. This means that we have $12 \times 2 \times 16 = 384$ strategy profiles.

F.) The smallest proper subgame belongs to the US and is the node that is marked with IV. After this, however, we see that there are no other proper subgames as they will either fail to have a unique initial node or they will “break” an information set. Hence, the only other subgame is the whole game itself. Therefore there are 2 subgames.
Problem 8. (8 points) In the game below determine the following:

A.) The total number of proper subgames.

B.) The total number of pure strategies for each player.

C.) The total number of pure strategy profiles.

Solution:

A.) The smallest proper subgames are those where player X gets to move. Traversing up the tree we see that the only other valid subgame is the whole game itself. However, because the whole game, while a subgame, is not a proper subgame so we do not count it. Therefore the game has 4 proper subgames.

B.) Every player in the game has exactly 2 available actions at each of their information sets.

<table>
<thead>
<tr>
<th>Player</th>
<th>Infosets</th>
<th>Strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>EH</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>JF</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>X</td>
<td>4</td>
<td>$2 \times 2 \times 2 \times 2 = 16$</td>
</tr>
<tr>
<td>Y</td>
<td>4</td>
<td>$2 \times 2 \times 2 \times 2 = 16$</td>
</tr>
<tr>
<td>Z</td>
<td>4</td>
<td>$2 \times 2 \times 2 \times 2 = 16$</td>
</tr>
</tbody>
</table>
C.) A pure strategy profile for the game will contain exactly 1 strategy for each player. In this case it will be a list of 5 strategies (a 5-tuple), \((s_{EH}, s_{JF}, s_{X}, s_{Y}, s_{Z})\). Since we have calculated the total number of strategies each players have we can count the total number of possible combinations of those strategies as \(2 \times 2 \times 16 \times 16 \times 16 = 16,384\).
Problem 9. (14 points) For the game below, find all pure and mixed strategy Nash equilibria.

Solution: The first step is to realize that this is a simultaneous game since both players make their strategy choices without knowledge of the other player’s strategy. We convert to a strategic form game and proceed from there. Each player has 1 information set with four actions meaning they each have 4 pure strategies and therefore we will have a total of $4 \times 4 = 16$ possible strategy profiles. Listing player 1’s strategies down the rows and player 2’s strategies across the columns we have the strategic form game below.

To reduce the difficulty in searching for both pure and mixed strategy Nash equilibria of this game we can take advantage of the assumption of rationality as common knowledge to perform iterated deletion of strictly dominated strategies (IDSDS) since these are never best responses and will not be played by rational players.

We note that $b$ is strictly dominated by $d$ for player 1 and that strategy $y$ is strictly dominated by $z$ for player 2. Hence we delete both strategies from the strategic form and we are left with
strategies and the fact that strategy $x$ is strictly dominated by $z$ for player 2 and will be deleted.

Because everyone knows that player 2 will not play $x$, its deletion causes strategy $a$ to become strictly dominated by $d$ for player 1 and we now have the reduced strategic form game after a couple rounds of IDSDS.

The remaining $2 \times 2$ game cannot be reduced any further by IDSDS since none of the strategies are strictly dominated. We now proceed to look for pure and mixed strategy Nash equilibria.

Using the best reply method to seek for pure strategy Nash equilibria we see that there are not any. We now seek for mixed strategy Nash equilibrium. Let $p$ be the probability that Player 2 plays $w$ and $q$ be the probability that Player 1 plays $c$. Rational players are only willing to randomize if and only if they are indifferent between the strategies over which they can randomize. Player 1 will only randomize if Player 2 behaves such
Player 2

\[
\begin{array}{cc}
  \text{w} & \text{z} \\
  \text{c} & (1, 3) & (4, 2) \\
  \text{d} & (5, 1) & (1, 4)
\end{array}
\]

Player 1

that the expected payoff from \( c \) is the same as \( d \).

\[
\begin{align*}
\text{Eu}_1(c) & : p + 4(1 - p) = 5p + (1 - p) \\
\text{Eu}_1(d) & : p + 4 - 4p = 5p - p + 1 \\
& \quad 4 - 3p = 4p + 1 \\
& \quad 3 = 7p \\
& \quad p^* = \frac{3}{7}
\end{align*}
\]

Player 2 is indifferent if

\[
\begin{align*}
\text{Eu}_2(w) & : 3q + (1 - q) = 2q + 4(1 - q) \\
\text{Eu}_2(z) & : 3q - q + 1 = 2q - 4q + 4 \\
& \quad 2q + 1 = 4 - 2q \\
& \quad 4q = 3 \\
& \quad q^* = \frac{3}{4}
\end{align*}
\]

Therefore the only Nash equilibrium of the game is in mixed strategies and is

\[
\left\{ \left( \frac{3}{4}c, \, \frac{1}{4}d \right), \, \left( \frac{3}{7}w, \, \frac{4}{7}z \right) \right\}
\]
Problem 10.  (14 points) Consider two players, Alice and Bob, who will play the ultimatum game with a pot worth $100. You and a friend are talking about how ultimatum games give power to the proposer because in subgame perfect equilibrium play the proposer is able to get the receiver to accept a smaller portion of the pot. Your friend then remarks,

“You know what the problem is? ... The proposer doesn’t face any competition. If we add another player and allow the receiver to decide who they want to play against, then the proposers will compete with each other and erase their proposer power.”

You consider this argument, but remain skeptical and relying on your ECNS 424 training you decide to test the argument with a simplified ultimatum game in which Alice, Bob and now Jill will play. Alice will choose whether to play the ultimatum game with Bob as the proposer, or with Jill as the proposer. If Alice rejects the proposer’s offer than the proposer gets 40 and Jill will only receive 20. The remaining 40 of the pot is surrendered to an outside party. The order of payoffs on the terminal nodes is (Alice, Bob, Jill).

![Game Diagram]

Find all subgame perfect Nash equilibria of the above game and then use the concept of sequential rationality to make an argument for why your friend is correct or incorrect about his hypothesis of competition improving the ultimatum problem.

Solution:
Using backward induction we see there are two pure strategy subgame perfect Nash equilibrium profiles

\[(\text{Bob}/\text{A}/\text{A}/\text{A}/\text{A}, 70-30, 70-30) \quad \text{(Jill}/\text{A}/\text{A}/\text{A}/\text{A}, 70-30, 70-30)\]

The payoff to Alice in both SPNE is 30, which is the same as she gets in SPNE play when she doesn’t have a choice between proposers. However, note that Alice is indifferent between her two SPNE strategies since she receives a payoff of 30 no matter who she picks to play against. Notice that Bob and Jill are not indifferent between the two SPNE. The left profile gives Bob 70 and while Jill gets 0. The right profile gives Bob 0 but Jill gets 70.

Suppose that Alice chooses Bob with probability \(b\). Then his expected payoff from his SPNE strategy of 70 – 30 is \(70b + 0(1 - b) = 70b\). If instead he was to play 50 – 50 his expected payoff would be \(50b + 0(1 - b) = 50b\). Since \(70b > 50b\) for all \(b\), Bob cannot be made indifferent between his strategies and will not randomize.

Similarly, Jill’s expected payoff from her SPNE strategy would be \(0b + 70(1 - b) = 70 - 70b\) and switching to 50 – 50 would yield \(0b + 50(1 - b)\). Again since \(70(1 - b) > 50(1 - b)\) for all \(b\), Jill will always want to play 70 – 30.
Problem 11. (14 points) Now reconsider the ultimatum game you just analyzed, but with some slightly different rules. A second friend approaches you and argues,

“*Alice would do best if Bob and Jill were forced to compete with each other more directly, by simultaneously making proposals to Alice, who then gets to decide which proposal to accept or to reject all of them.*”

Is your other friend correct? Again relying on your training you build a simple model of this process where Bob and Jill decide on their proposal simultaneously and then present it to Alice. Alice observes the proposals and then decides to accept Bob’s proposal ($A_B$) or to accept Jill’s proposal ($A_J$). If Alice accepts, then Alice and whoever’s proposal she accepted receive a payoff equal to their portion. If Alice rejects both proposals, then she gets 10 while Bob and Jill split the remaining 50, each receiving 25.

Begin with backward induction for Alice’s choices and consider the possible subgame perfect Nash equilibrium outcomes. Determine whether Alice will be better off than she was in the previous extended ultimatum game from problem 8. Finally, provide a short rationalization for why these results differ from Problem 8 (if they are different) or why they are the same as Problem 8 (if the same).

**Solution:** We begin by using backward induction on the smallest proper subgames which are all of Alice’s information sets. The next largest subgame is the entire game itself and we will have a simultaneous game between Bob and Jill.
When Bob and Jill make the same proposals, Alice has multiple best responses in the subgames. In terms of her sequential rationality on those subgames she is indifferent between selecting either offer and is therefore technically willing to randomize whose offer she accepts. Alice only has 4 strategies that are consistent with sequential rationality.

\[ s_1 = A_B/A_B/A_J/A_B \quad s_2 = A_B/A_B/A_J/A_J \]
\[ s_3 = A_J/A_B/A_J/A_B \quad s_4 = A_J/A_B/A_J/A_J \]

At this point we can begin to answer the question of whether Alice is better off in this game. From the backward induction of the subgames corresponding to Alice’s first three (left-to-right) information sets we see that Alice is able to guarantee herself a payoff of 50. The previous game had two subgame perfect Nash equilibria and in each of those equilibria Alice had to accept a 70-30 split. However, in all of the subgames here Alice does better accept for the far right decision node in which both Bob and Jill make 70-30 proposals. This means that the current version of the proposal game allows Alice to do better as long as Bob and Jill both offering 70-30 splits cannot be sustained as a Nash equilibrium.

To test these conditions we look at the four relevant strategy profiles involving Bob and Jill choosing 70-30 and Alice playing one of \( s_1, s_2, s_3, s_4 \). When Alice plays \( s_1 \) then Bob’s offer will be accepted giving him 70 while Jill will get 0. Because \( s_1 \) specifies that Alice accept Jill’s offer if Jill offers 50-50 while Bob proposes 70-30, Jill can do strictly better (50 vs 0) if she deviates. Hence \((s_1, 70 - 30, 70 - 30)\) is not a Nash equilibrium of the game and is therefore not a subgame perfect Nash equilibrium. This same logic will rule out the profile \((s_3, 70 - 30, 70 - 30)\).

Now consider \((s_2, 70 - 30, 70 - 30)\) which specifies that Alice will accept Jill’s offer when Bob and Jill play 70-30. Now Jill has no incentive to deviate since she gets her best payoff. If Bob deviates to 50-50 while Jill plays 70-30, then Alice’s strategy (and sequential rationality) dictate she accepts Bob’s 50-50 offer. Since Bob does strictly better (50 vs 0) Bob will unilaterally deviate; ruling the profile out as a Nash equilibrium. Finally we have the profile \((s_4, 70 - 30, 70 - 30)\) which again specifies Jill’s offer be accepted. As before, Bob can do better by deviating to 50-50, so this also is not a Nash equilibrium.
Because no such strategy profile can be a Nash equilibrium, it cannot be subgame perfect and as a consequence there are no subgame perfect pure strategy Nash equilibrium that provide a payoff to Alice less than 50. So she does indeed to better.
Problem 12. (15 points) Consider two firms who have similar, but differentiated products. They compete with each other in price and are both capable of advertising to their customers. The timing of the game is as follows.

First, both firms choose an advertising level \( A_i \in [0, \infty) \) for all \( i = 1, 2 \) before observing demand. Advertising is quite costly and the cost is increasing and concave such that the cost of choosing level \( A \) has cost \( c(A) = A^2 \). Second, after both firms choose their advertising level and pay advertising fees, both firms observe the levels chosen and the resulting differentiated product’s demand curves:

\[
q_1(p_1, p_2) = A_1 - 2p_1 + p_2 \\
q_2(p_1, p_2) = A_2 - 2p_2 + p_1
\]

Third, the firms will then simultaneously choose their prices \( p_i \in [0, \infty) \) for all \( i = 1, 2 \) to maximize their profit. Assume both firms have an identical marginal cost of $10 per unit of product.

Find the subgame perfect Nash equilibrium values for \( A_1, A_2, p_1 \) and \( p_2 \).

Solution: Note that this game is a sequence of two simultaneous games. Because firms will observe each other’s choice of \( A \) before choosing their prices, they will take \( A \) as fixed in the second stage of the game. Using backward induction we will first analyze the Differentiated Bertrand pricing game the firm’s engage in the second stage. The firm’s profit functions are

\[
\pi_1 = (A_1 - 2p_1 + p_2)(p_1 - 10) = A_1p_1 - 2p_1^2 + p_2p_1 + 20p_1 - 10A_1 - 10p_2 \\
\pi_2 = (A_2 - 2p_2 + p_1)(p_2 - 10) = A_2p_2 - 2p_2^2 + p_1p_2 + 20p_2 - 10A_2 - 10p_1
\]

To find the Nash equilibrium of this subgame, we will derive the best response functions for each of the firms. Taking derivatives with respect to \( p_i \) and setting the equation equal to zero we have,

\[
\frac{\partial \pi_1}{\partial p_1} = A_1 - 4p_1 + p_2 + 20 = 0 \\
\frac{\partial \pi_2}{\partial p_2} = A_2 - 4p_2 + p_1 + 20 = 0
\]

We now find each firm’s best response functions by solving each equation for \( p_i^* \) in terms of \( p_j \). We get,

\[
p_1^*(A_1, p_2) = \frac{A_1 + p_2 + 20}{4} \\
p_2^*(A_2, p_1) = \frac{A_2 + p_1 + 20}{4}
\]
Plugging $p_2^*$ into $p_1^*$ we have

\[ p_1^* = \frac{A_1}{4} + \frac{A_2 + p_1^* + 20}{16} + 5 \]
\[ = \frac{4A_1 + A_2 + p_1^*}{16} + \frac{20}{16} + 5 \]
\[ = \frac{4A_1 + A_2 + p_1^*}{16} + 5 \frac{5}{4} + 5 \]

\[ p_1^* - \frac{p_1^*}{16} = \frac{4A_1 + A_2}{16} + 5 \frac{5}{4} + 5 \]
\[ \frac{15}{16}p_1^* = 4A_1 + A_2 + 100 \]
\[ 15p_1^* = 4A_1 + A_2 + 100 \]
\[ p_1^* = \frac{4A_1 + A_2 + 100}{15} \]

Now that we have the value of $p_1^*$ we plug it into $p_2^*$

\[ p_2^* = \frac{A_2}{4} + \frac{14A_1 + A_2 + 100}{15} + 5 \]
\[ = \frac{A_2}{4} + \frac{A_2}{60} + \frac{A_1}{15} + \frac{25}{15} + \frac{75}{15} \]
\[ = \frac{15A_2 + A_2}{60} + \frac{A_1}{15} + \frac{100}{15} \]
\[ p_2^* = \frac{4A_2 + A_1 + 100}{15} \]

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**Problem 13.** (15 points) Consider a game in which $M$ buyers and $N$ sellers choose to either participate in one of two online auction sites, eBay and Yahoo!, or choose to participate in neither. Buyers and sellers choose one auction site to use $1 - eBay$ and $2 - Yahoo!$ so that the strategy set for every buyer and seller is $\{1, 2\}$. Each auction site has a price that they charge buyers $p^b_i$ and a price they charge sellers $p^s_i$ where $i$ indexes the auction site (i.e, $1 - eBay$ and $2 - Yahoo!$).

Suppose $s_i$ be the number of sellers at site $i$ and let $b_i$ be the number of buyers at site $i$. The payoff to a buyer at site $i$ is

$$10s_i - p^b_i$$

and the payoff to a seller at site $i$ is

$$5b_i - p^s_i$$

**Find all pure strategy Nash equilibria of the game.**

**Solution:** Two symmetric strategy profiles - all buyers and sellers choose eBay or all buyers and sellers choose Yahoo!. Let's start with the former. If all $M$ buyers and $N$ sellers choose 1 then each buyer receives a payoff of

$$10N - p^b_1$$

while the payoff to each seller is

$$5M - p^s_1$$

Now consider any buyer who might wish to switch auction sites. The sole buyer at site 2 is going to have payoff $-p^b_2$ which is the price of participating at auction site 2. The same argument follows any buyers considering unilateral deviation to site 2 they would experience payoff $-p^s_2$. If we assume that non-participation is an option and that it is ranked as a payoff of 0 then Nash equilibrium requires the following to hold,

$$10N - p^b_1 \geq 0$$

$$5M - p^s_1 \geq 0$$

which it does. So the symmetric strategy profile where all $M$ buyers and $N$ sellers choose to participate on eBay is a Nash equilibrium. An identical analysis of the symmetric profile where all $M$ buyers and $N$ sellers choose Yahoo! will show that it also is a Nash equilibrium.

Is there a Nash equilibrium where some buyers and sellers are on eBay and some buyers and sellers are on Yahoo!?

- $b_1$ buyers go to site 1 (eBay) and $M - b_1$ buyers go to site 2 (Yahoo!) and $1 < b_1 < M$.
- Let $s_1$ sellers be at site 1 and $s_2$ sellers be at site 2 such that $s_1 + s_2 = N$.

$10 \times s_1 - p^b_1$ Payoff to buyer at site 1.

$10 \times s_2 - p^b_2$ Payoff to buyer at site 2.
For some buyers to go to each site, they cannot do strictly better by switching to the site they are not currently on. Intuitively, they need to be getting the same payoff (at least close to it) from each site.

\[
10 \times s_1 - p_b^1 = 10 \times (N - s_1) - p_b^2
\]

\[
s_1^* = \left( \frac{1}{20} \right) (10N + p_b^1 - p_b^2)
\]

So there needs to be \( s_1^* \) sellers at site 1 and \( 1 - s_1^* \) sellers at site 2 to give buyers equal payoffs. It is also required that buyers prefer participation at some site to no participation on any site. In other words,

\[
10 \times s_1 - p_b^1 \geq 0
\]

Substitute \( s_1^* \) in for \( s_1 \) in the above condition

\[
10 \left( \left( \frac{1}{20} \right) (10N + p_b^1 - p_b^2) \right) - p_b^1 \geq 0
\]

\[
10N \geq p_b^1 + p_b^2
\]

Similarly, sellers will participate at different sites as long as the payoffs to a seller at both sites is the same and better than not participating at all. In other words,

\[
5b_1 - p_s^1 = 5b_2 - p_s^2
\]

\[
5b_1 - p_s^1 \geq 0
\]

Solving the equation for \( b_1 \) (set \( b_2 = M - b_1 \)) and substituting into the inequality condition

\[
5 \left[ \left( \frac{1}{10} \right) (5M + p_s^1 - p_s^2) \right] - p_s^1 \geq 0
\]

which implies that \( 5M \geq p_s^1 + p_s^2 \)

Finally we have two conditions that can allow an asymmetric Nash equilibrium to exist.

1. \( p_b^1 + p_b^2 \leq 10N \)
2. \( p_s^1 + p_s^2 \leq 5M \)

These conditions say that as long as prices charged to buyers and sellers at both sites are sufficiently low, then we could arrange buyers and sellers between both sites in such a way that no one can do strictly better by unilaterally deviating to the other site.