Optimization:

For each problem below, determine the optimal output level for each firm given the inverse demand function and marginal cost.

1. $p = 500 - q \quad MC = 50$

   Setting up the profit maximization problem,

   $$\max_{q} \ (500 - q)q - 50q$$

   Calculating a first-order condition,

   $$\frac{d\pi}{dq} = 500 - 2q - 50 = 0$$

   Solving for q,

   $$q^* = \frac{450}{2} = 225$$

2. $p = 1000 - 2q \quad MC = 200$

   Setting up the profit maximization problem,

   $$\max_{q} \ (1000 - 2q)q - 200q$$

   Calculating a first-order condition,

   $$\frac{d\pi}{dq} = 1000 - 4q - 200 = 0$$

   Solving for q,

   $$q^* = \frac{800}{4} = 200$$

3. $p = 300 - q_1 - q_2 \quad MC = 60$

   Setting up the profit maximization problem for firm 1,

   $$\max_{q_1} \ (300 - q_1 - q_2)q_1 - 60q_1$$

Calculating a first-order condition for firm 1,

$$\frac{\partial \pi_1}{\partial q_1} = 300 - 2q_1 - q_2 - 60 = 0$$

Setting up the profit maximization problem for firm 2,

$$\max_{q_2} \ (300 - q_1 - q_2)q_2 - 60q_2$$

Calculating a first-order condition for firm 2,

$$\frac{\partial \pi_2}{\partial q_2} = 300 - q_1 - 2q_2 - 60 = 0$$

Note here that we can invoke symmetry; let $q = q_1 = q_2$,

$$300 - 2q - q - 60 = 0$$
$$3q = 240$$
$$q^* = q_1^* = q_2^* = 80$$

4. $p_1 = 400 - 2q_1 - q_2 \quad MC_1 = 0$
   $p_2 = 400 - q_1 - 2q_2 \quad MC_2 = 0$

Setting up the profit maximization problem for firm 1,

$$\max_{q_1} \ (400 - 2q_1 - q_2)q_1$$

Calculating a first-order condition for firm 1,

$$\frac{\partial \pi_1}{\partial q_1} = 400 - 4q_1 - q_2 = 0$$

Setting up the profit maximization problem for firm 2,

$$\max_{q_2} \ (400 - q_1 - 2q_2)q_2$$

Calculating a first-order condition for firm 2,

$$\frac{\partial \pi_2}{\partial q_2} = 400 - q_1 - 4q_2 = 0$$

Note here that we can invoke symmetry; let $q = q_1 = q_2$,

$$400 - 4q - q = 0$$

$$5q = 400$$
$$q^* = q_1^* = q_2^* = 80$$

For each problem below, calculate the optimal bundle of goods x and y given the utility function and budget constraint.

5. $u(x, y) = x^{0.4}y^{0.6} \quad 2x + 3y \leq 50$

Setting up the utility maximization problem,

$$\max_{x,y,\lambda} x^{0.4}y^{0.6} + \lambda(50 - 2x - 3y)$$

Calculating first-order conditions,

$$\frac{\partial \mathcal{L}}{\partial x} = 0.4x^{-0.6}y^{0.6} - 2\lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = 0.6x^{0.4}y^{-0.4} - 3\lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 50 - 2x - 3y = 0$$

Rearranging then dividing the first equation by the second,

$$\frac{0.4x^{-0.6}y^{0.6}}{0.6x^{0.4}y^{-0.4}} = \frac{2\lambda}{3\lambda}$$
$$\frac{2y}{3x} = \frac{2}{3}$$
$$y = x$$

Substituting this into the third equation,

$$50 - 2x - 3x = 0$$
$$5x = 50$$
$$x^* = 10$$

From the tangency condition,

$$y^* = x^* = 10$$

Checking the Lagrange multiplier,

$$\lambda^* = 0.4(x^*)^{-0.6}(y^*)^{0.6} = 0.4$$

Since the Lagrange multiplier is positive, the constraint is binding and we have our equilibrium.

6. $u(x, y) = -(x-3)^3(y-2)^2 \quad x + y \leq 10$

Setting up the utility maximization problem,

$$\max_{x,y,\lambda} -(x-3)^3(y-2)^2 + \lambda(10 - x - y)$$

Calculating first-order conditions,

$$\frac{\partial \mathcal{L}}{\partial x} = -3(x-3)^2(y-2)^2 - \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = -2(x-3)^3(y-2) - \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 10 - x - y = 0$$

Rearranging then dividing the first equation by the second,

$$\frac{-3(x-3)^2(y-2)^2}{-2(x-3)^3(y-2)} = \frac{\lambda}{\lambda}$$
$$\frac{3(y-2)}{2(x-3)} = 1$$
$$3y - 6 = 2x - 6$$
$$3y = 2x$$

Substituting this into the third equation,

$$10 - x - \frac{2}{3}x = 0$$
$$\frac{5}{3}x = 10$$
$$x^* = 6$$

From the tangency condition,

$$y^* = \frac{2}{3}x^* = 4$$

Checking the Lagrange multiplier,

$$\lambda^* = -3(x^* - 3)^2(y^* - 2)^2 = -108$$

Since the Lagrange multiplier is negative, the constraint does not bind and we can discard it. Setting up a new utility maximization problem,

$$\max_{x,y} -(x-3)^3(y-2)^2$$

Calculating first-order conditions,

$$\frac{\partial \mathcal{L}}{\partial x} = -3(x-3)^2(y-2)^2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = -2(x-3)^3(y-2) = 0$$

This is just two sets of one equation and one unknown. The solutions to these equations are $x^* = 3$ and $y^* = 2$.