Constrained Optimization

Basic Math for Economics – Refresher

Introduction

- With an introduction to constraints out of the way, we are ready to talk about constrained optimization.
  - Recall that in these situations, we are trying to optimize some objective function subject to one or more constraints.
    - These constraints may not even be binding, so they must be checked.
    - Otherwise, we make use of the Lagrangian technique and use our first and second-order conditions as before.
    - Let's look at a couple of examples.

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- Consider a situation where a consumer is deciding how much of goods x and y to purchase and receives utility from these two goods as follows:

$$u(x, y) = x^{0.5}y^{0.5}$$

- In addition, the consumer has an income level of I and faces prices p_x and p_y for goods x and y, respectively.
  - Find the consumer's optimal bundle of goods x and y as a function of income I, and the prices of each good.
    - Let's set this optimization problem up.

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- This is a utility maximization problem for choice variables x and y. In addition, our objective function is the utility function.
  - Adding all of this to our optimization problem, we have

$$\max_{x,y} \; x^{0.5}y^{0.5}$$

- We also need to consider our budget constraint.
  - The total amount that our consumer spends on goods x and y cannot exceed their income, I, i.e.,

$$p_x x + p_y y \leq I$$

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$$p_x x + p_y y \leq I$$

- Rearranging our constraint such that it is greater than or equal to zero,

$$I - p_x x - p_y y \geq 0$$

- Now we assemble our Lagrangian by inserting the constraint along with our objective function (don't forget to include a Lagrange multiplier).

$$\max_{x,y,\lambda} x^{0.5}y^{0.5} + \lambda\left(I - p_x x - p_y y\right)$$

  - We are now ready to calculate first-order conditions.

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$$\max_{x,y,\lambda} \; x^{0.5}y^{0.5} + \lambda\left(I - p_x x - p_y y\right)$$

- With three choice variables, we need three first-order conditions.
  - Taking the derivative of our Lagrangian for each of our choice variables and setting them equal to zero yields,

$$\frac{\partial \mathcal{L}}{\partial x} = 0.5x^{-0.5}y^{0.5} - \lambda p_x = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 0.5x^{0.5}y^{-0.5} - \lambda p_y = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = I - p_x x - p_y y = 0$$

    - This is just three equations and three unknowns, which we can solve.

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$$\frac{\partial \mathcal{L}}{\partial x} = 0.5x^{-0.5}y^{0.5} - \lambda p_x = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = 0.5x^{0.5}y^{-0.5} - \lambda p_y = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = I - p_x x - p_y y = 0$$

- Let's focus on the first two equations. Rearranging them, we have

$$0.5x^{-0.5}y^{0.5} = \lambda p_x$$
$$0.5x^{0.5}y^{-0.5} = \lambda p_y$$

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$$0.5x^{-0.5}y^{0.5} = \lambda p_x$$
$$0.5x^{0.5}y^{-0.5} = \lambda p_y$$

- Next, we divide the first equation by the second, which cancels out λ.

$$\frac{0.5x^{-0.5}y^{0.5}}{0.5x^{0.5}y^{-0.5}} = \frac{\lambda p_X}{\lambda p_y} = \frac{p_x}{p_y}$$

- We can then simplify this expression to obtain

$$\frac{y}{x} = \frac{p_x}{p_y}$$

$$p_X x = p_y y$$

  - This is known as a tangency condition.

Constrained Optimization

$$p_X x = p_y y$$

$$I - p_x x - p_y y = 0$$

- Now we use the tangency condition along with our final first-order condition. Substituting the tangency condition in,

$$I - p_x x - \underbrace{p_x x}_{p_y y} = 0$$

- Rearranging terms, we have

$$2p_x x = I$$

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$$2p_x x = I$$

- Next, we divide both sides by $2p_x$ to obtain our equilibrium value of good x,

$$x^* = \frac{I}{2p_x}$$

- From here, we can use the tangency condition to solve for our equilibrium value of good y,

$$p_y y = p_x x^* = p_x \frac{I}{2p_x}$$

$$y^* = \frac{I}{2p_y}$$

  - And we have our equilibrium bundle!

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- Lastly, we need to check the value of λ to ensure that the constraint binds.
  - This can be done with either of the first two first-order conditions. Using the first one,

$$0.5x^{-0.5}y^{0.5} - \lambda p_x = 0$$

  - Rearranging terms and plugging in our equilibria,

$$\lambda = \frac{1}{2p_x}\left(\frac{I}{2p_x}\right)^{-0.5}\left(\frac{I}{2p_y}\right)^{0.5} = \left(\frac{1}{2p_x p_y}\right)^{0.5}$$

    - Since prices are positive, we can assume that $\lambda > 0$ and our constraint binds.

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- We can continue and check our second-order conditions as well.
  - I will leave that for you to try on your own. You'll find that we do, in fact, have a maximum.
- Let's look at one more example, with a different utility function. This time, the consumer receive utility from goods x and y of

$$u(x, y) = -(x-2)^2(y-3)^2$$

  - In addition, the consumer has an income level of $I = 11$, and faces prices of $p_x = p_y = 1$.

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- Once again, we can set up their Lagrangian by adding their constraint to their optimization problem to obtain,

$$\max_{x,y,\lambda} \; -(x-2)^2(y-3)^2 + \lambda(11-x-y)$$

- Calculating first-order conditions,

$$\frac{\partial \mathcal{L}}{\partial x} = -2(x-2)(y-3)^2 - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = -2(x-2)^2(y-3) - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 11 - x - y = 0$$

  - Once again, this is just three equations and three unknowns.

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$$\frac{\partial \mathcal{L}}{\partial x} = -2(x-2)(y-3)^2 - \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = -2(x-2)^2(y-3) - \lambda = 0$$

- Rearranging and combining the first two equations, we have,

$$\frac{-2(x-2)(y-3)^2}{-2(x-2)^2(y-3)} = \frac{\lambda}{\lambda} = 1$$

  - Rearranging once again, we obtain our tangency condition,

$$\frac{y-3}{x-2} = 1$$
$$y = x + 1$$

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$$y = x + 1$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 11 - x - y = 0$$

- Substituting our tangency condition into our third first-order condition,

$$11 - x - \underbrace{(x + 1)}_{y} = 0$$

- Solving this expression for x, we obtain our equilibrium quantity,

$$x^* = 5$$

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- From our tangency condition we obtain our equilibrium quantity of y,

$$y^* = x^* + 1 = 6$$

- Finally, we check to see if our constraint binds. Using the first first-order condition and solving it for λ,

$$\lambda = -2(x-2)(y-3)^2 = -2(5-2)(6-3)^2 = -54$$

- This time the Lagrange multiplier comes out negative!
  - This implies that our constrain does not bind, and is in fact pushing our equilibrium away from its maximum point.
    - Thus, our equilibrium is **not** utility maximizing.

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- To fix this, we simply throw out the constraint and treat this problem as an unconstrained optimization problem.
  - Work through it, and you'll find that the consumer optimizes by selecting $x^* = 2$ and $y^* = 3$.
    - In this special kind of utility function, the consumer has a bliss point at exactly this level of consumption.
    - They would actually rather keep some of their money even though they could purchase more.
      - This is why it is important to check the values of the Lagrange multiplier.