Constrained Optimization

Basic Math for Economics – Refresher
With an introduction to constraints out of the way, we are ready to talk about constrained optimization.

Recall that in these situations, we are trying to optimize some objective function subject to one or more constraints.

These constraints may not even be binding, so they must be checked.

Otherwise, we make use of the Lagrangian technique and use our first and second-order conditions as before.

Let’s look at a couple of examples.
Consider a situation where a consumer is deciding how much of goods $x$ and $y$ to purchase and receives utility from these two goods as follows:

$$u(x, y) = x^{0.5} y^{0.5}$$

In addition, the consumer has an income level of $I$ and faces prices $p_x$ and $p_y$ for goods $x$ and $y$, respectively.

Find the consumer’s optimal bundle of goods $x$ and $y$ as a function of income $I$, and the prices of each good.

Let’s set this optimization problem up.
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- This is a utility maximization problem for choice variables $x$ and $y$. In addition, our objective function is the utility function.
  - Adding all of this to our optimization problem, we have
    $$\max_{x,y} x^{0.5} y^{0.5}$$
- We also need to consider our budget constraint.
  - The total amount that our consumer spends on goods $x$ and $y$ cannot exceed their income, $I$, i.e.,
    $$p_x x + p_y y \leq I$$
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\[ p_x x + p_y y \leq I \]

- Rearranging our constraint such that it is greater than or equal to zero,
  \[ I - p_x x - p_y y \geq 0 \]

- Now we assemble our Lagrangian by inserting the constraint along with our objective function (don’t forget to include a Lagrange multiplier).
  \[ \max_{x, y, \lambda} x^{0.5} y^{0.5} + \lambda(I - p_x x - p_y y) \]

- We are now ready to calculate first-order conditions.
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$$\max_{x,y,\lambda} x^{0.5}y^{0.5} + \lambda(I - px - py)$$

○ With three choice variables, we need three first-order conditions.
  ○ Taking the derivative of our Lagrangian for each of our choice
    variables and setting them equal to zero yields,
    $$\frac{\partial L}{\partial x} = 0.5x^{-0.5}y^{0.5} - \lambda p_x = 0$$
    $$\frac{\partial L}{\partial y} = 0.5x^{0.5}y^{-0.5} - \lambda p_y = 0$$
    $$\frac{\partial L}{\partial \lambda} = I - px - py y = 0$$

○ This is just three equations and three unknowns, which we can
  solve.
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\[
\frac{\partial L}{\partial x} = 0.5x^{-0.5}y^{0.5} - \lambda p_x = 0 \\
\frac{\partial L}{\partial y} = 0.5x^{0.5}y^{-0.5} - \lambda p_y = 0 \\
\frac{\partial L}{\partial \lambda} = I - px x - py y = 0
\]

Let's focus on the first two equations. Rearranging them, we have

\[
0.5x^{-0.5}y^{0.5} = \lambda p_x \\
0.5x^{0.5}y^{-0.5} = \lambda p_y
\]
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\[
0.5x^{-0.5}y^{0.5} = \lambda p_x \\
0.5x^{0.5}y^{-0.5} = \lambda p_y
\]

- Next, we divide the first equation by the second, which cancels out \( \lambda \).

\[
\frac{0.5x^{-0.5}y^{0.5}}{0.5x^{0.5}y^{-0.5}} = \frac{\lambda p_x}{\lambda p_y} = \frac{p_x}{p_y}
\]

- We can then simplify this expression to obtain

\[
\frac{y}{x} = \frac{p_x}{p_y}
\]

\[
p_x x = p_y y
\]

- This is known as a tangency condition.
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\[ p_x x = p_y y \]
\[ I - p_x x - p_y y = 0 \]

- Now we use the tangency condition along with our final first-order condition. Substituting the tangency condition in,

\[ I - p_x x - \frac{p_x x}{p_y y} = 0 \]

- Rearranging terms, we have

\[ 2p_x x = I \]
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\[ 2p_x x = I \]

- Next, we divide both sides by \( 2p_x \) to obtain our equilibrium value of good \( x \),

\[ x^* = \frac{I}{2p_x} \]

- From here, we can use the tangency condition to solve for our equilibrium value of good \( y \),

\[ p_y y = p_x x^* = p_x \frac{I}{2p_x} \]

\[ y^* = \frac{I}{2p_y} \]

- And we have our equilibrium bundle!
Lastly, we need to check the value of $\lambda$ to ensure that the constraint binds.

This can be done with either of the first two first-order conditions. Using the first one,

$$0.5x^{-0.5}y^{0.5} - \lambda p_x = 0$$

Rearranging terms and plugging in our equilibria,

$$\lambda = \frac{1}{2p_x} \left( \frac{I}{2p_x} \right)^{-0.5} \left( \frac{I}{2p_y} \right)^{0.5} = \left( \frac{1}{2p_x p_y} \right)^{0.5}$$

Since prices are positive, we can assume that $\lambda > 0$ and our constraint binds.
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- We can continue and check our second-order conditions as well.
  - I will leave that for you to try on your own. You’ll find that we do, in fact, have a maximum.

- Let’s look at one more example, with a different utility function. This time, the consumer receive utility from goods $x$ and $y$ of

$$u(x, y) = -(x - 2)^2(y - 3)^2$$

- In addition, the consumer has an income level of $I = 11$, and faces prices of $p_x = p_y = 1$. 
Once again, we can set up their Lagrangian by adding their constraint to their optimization problem to obtain,

$$\max_{x, y, \lambda} - (x - 2)^2 (y - 3)^2 + \lambda (11 - x - y)$$

Calculating first-order conditions,

$$\frac{\partial L}{\partial x} = -2(x - 2)(y - 3)^2 - \lambda = 0$$

$$\frac{\partial L}{\partial y} = -2(x - 2)^2 (y - 3) - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 11 - x - y = 0$$

Once again, this is just three equations and three unknowns.
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\[
\frac{\partial L}{\partial x} = -2(x - 2)(y - 3)^2 - \lambda = 0
\]
\[
\frac{\partial L}{\partial y} = -2(x - 2)^2(y - 3) - \lambda = 0
\]

○ Rearranging and combining the first two equations, we have,

\[
\frac{-2(x - 2)(y - 3)^2}{-2(x - 2)^2(y - 3)} = \frac{\lambda}{\lambda} = 1
\]

○ Rearranging once again, we obtain our tangency condition,

\[
\frac{y - 3}{x - 2} = 1
\]

\[
y = x + 1
\]
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\[ y = x + 1 \]

\[ \frac{\partial L}{\partial \lambda} = 11 - x - y = 0 \]

- Substituting our tangency condition into our third first-order condition,

\[ 11 - x - (x + 1) = 0 \]

- Solving this expression for \( x \), we obtain our equilibrium quantity,

\[ x^* = 5 \]
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- From our tangency condition we obtain our equilibrium quantity of $y$,

$$y^* = x^* + 1 = 6$$

- Finally, we check to see if our constraint binds. Using the first first-order condition and solving it for $\lambda$,

$$\lambda = -2(x - 2)(y - 3)^2 = -2(5 - 2)(6 - 3)^2 = -54$$

- This time the Lagrange multiplier comes out negative!

  - This implies that our constrain does not bind, and is in fact pushing our equilibrium away from its maximum point.

  - Thus, our equilibrium is **not** utility maximizing.
To fix this, we simply throw out the constraint and treat this problem as an unconstrained optimization problem.

Work through it, and you’ll find that the consumer optimizes by selecting $x^* = 2$ and $y^* = 3$.

In this special kind of utility function, the consumer has a bliss point at exactly this level of consumption.

They would actually rather keep some of their money even though they could purchase more.

This is why it is important to check the values of the Lagrange multiplier.