Unconstrained Optimization: Multiple Variables

Basic Math for Economics – Refresher

Introduction

- While working with just one variable is simple, it's rare in economics.
  - Often, we must work with many variables all at the same time.
  - This complicates our optimization slightly, but the same rules still apply.
- Let's take a look at another unconstrained optimization problem, but this time through the lens of a Cournot duopoly.
  - Recall that in a Cournot duopoly, two firms simultaneously choose their quantities, but face the same market price.

Unconstrained Optimization

- Let's use the same market as last time, but rather than there being a monopolist, now there are two firms and they compete in quantities.
  - Both firms face the inverse demand function
  $$p = 250 - 2q_1 - 2q_2$$
  and face constant marginal costs of production of $c = 50$.
- As before, we will set up our problem and then optimize.
  - Note that we actually have two optimization problems here.
    - Each firm is maximizing their profits separately, so we need to treat them as separate optimization decisions.

Unconstrained Optimization

- Let's start with firm 1.
  - They want to maximize their profits, and the only thing they can choose is their own quantity, q_1. Thus, we have,

$$\max_{q_1}$$

  - Next, they want to maximize their own profits (they don't care about firm 2's profits at all), but their inverse demand function is also a function of firm 2's quantity, so we must include it.

$$\max_{q_1} \; (250 - 2q_1 - 2q_2)q_1 - 50q_1$$

Unconstrained Optimization

$$\max_{q_1} \; (250 - 2q_1 - 2q_2)q_1 - 50q_1$$

- From here, we can calculate a first-order condition as before (remember to treat q_2 as a constant),

$$\frac{\partial \pi_1}{\partial q_1} = 250 - 4q_1 - 2q_2 - 50 = 0$$

- We can perform the same analysis for firm 2, obtaining the following optimization problem,

$$\max_{q_2} \; (250 - 2q_1 - 2q_2)q_2 - 50q_2$$

and first-order condition,

$$\frac{\partial \pi_2}{\partial q_2} = 250 - 2q_1 - 4q_2 - 50 = 0$$

Unconstrained Optimization

$$\frac{\partial \pi_1}{\partial q_1} = 250 - 4q_1 - 2q_2 - 50 = 0$$
$$\frac{\partial \pi_2}{\partial q_2} = 250 - 2q_1 - 4q_2 - 50 = 0$$

- This is just a system of two equations and two unknowns and fairly easy to solve.
  - Doing so yields $q_1^* = q_2^* = 33.3$, which is notable smaller than what a monopolist produces (which is expected).
    - We still need to check whether this is a maximum, however.

Unconstrained Optimization

$$\frac{\partial \pi_1}{\partial q_1} = 250 - 4q_1 - 2q_2 - 50 = 0$$
$$\frac{\partial \pi_2}{\partial q_2} = 250 - 2q_1 - 4q_2 - 50 = 0$$

- This is a bit harder, and requires the use of a matrix called the Hessian.
  - To calculate the Hessian, we need to take a derivative of each of our first-order conditions for each of our choice variables (4 derivatives total in this case).
    - The first row and first column of our Hessian is the derivative of our first first-order condition for our first variable.
    - The first row and second column of our Hessian is the derivative of our first first-order condition for our second variable.

Unconstrained Optimization

- Calculating these derivatives,

$$H = \begin{bmatrix} \frac{\partial^2 \pi_1}{\partial q_1^2} = -4 & \frac{\partial^2 \pi_1}{\partial q_1 \partial q_2} = -2 \\ \frac{\partial^2 \pi_2}{\partial q_2 \partial q_1} = -2 & \frac{\partial^2 \pi_2}{\partial q_2^2} = -4 \end{bmatrix}$$

- Or, reducing this matrix a little bit, we have

$$H = \begin{bmatrix} -4 & -2 \\ -2 & -4 \end{bmatrix}$$

  - Our problem is maximized if this matrix is negative semidefinite.

Unconstrained Optimization

$$H = \begin{bmatrix} -4 & -2 \\ -2 & -4 \end{bmatrix}$$

- Negative semidefinite?
  - Leaving out most of the math, it means that certain elements of the Hessian must take certain values.
    - For a 2x2 matrix, if all the elements along the trace are less than or equal to zero and the determinant is positive, then the matrix is negative semidefinite.
      - First, let's check the trace. Both elements along the trace are equal to -4, so they satisfy the requirement that they are less than or equal to zero.

Unconstrained Optimization

$$H = \begin{bmatrix} -4 & -2 \\ -2 & -4 \end{bmatrix}$$

- Now, let's calculate the determinant.

$$|H| = -4(-4) - (-2)(-2) = 16 - 4 = 12$$

  - Since the determinant of the Hessian is positive, that along with all the elements of our trace being negative means that our matrix is negative semidefinite and we have a maximum rather than a minimum.
    - It's useful to check this when dealing with less behaved functional forms.

Unconstrained Optimization

- If you are looking for a minimum, rather than a maximum, the Hessian needs to be positive semidefinite.
  - For a 2x2 Hessian, all the elements along the trace must be greater than or equal to zero while the determinant is still positive.
- If we move beyond a 2x2 Hessian (i.e., we have more than two choice variables), this becomes more complicated.
  - Our principal minors must have alternating signs based upon which minor they are.
    - A linear algebra course would be helpful for this.