Unconstrained Optimization: Single Variable

Basic Math for Economics – Refresher

Introduction

- In both micro and macroeconomic contexts, optimization is a frequently relied upon tool.
  - We use it to maximize profits, minimize costs, etc.
- We break optimization down into two types:
  - Unconstrained Optimization deals with situations where we seek to either maximize or minimize one or several variables without any restrictions on the values they can take.
  - Constrained Optimization, on the other hand, imposes limits on the values that our variables can take.
    - Things like budget constraints, production functions, etc.

Unconstrained Optimization

- Let's review unconstrained optimization by looking at the profit maximization function.
  - Suppose a monopolist served a market that faced the inverse demand function of $p = 250 - 2q$ and a constant marginal cost of production of $c = 50$.
    - What value of q maximizes the monopolist's profits? What is the corresponding price and profit level?
  - We already know that the monopolist can set his marginal revenue equal to his marginal costs and determine his optimal quantity that way.
    - This time, let's set it up using an optimization problem.

Unconstrained Optimization

- To properly set up an optimization problem, we need a few elements:
  - First, we need to define the problem. This is a maximization problem, so let's start by writing that.

$$\max$$

  - Next, we need to list the choice variables; the ones that we are optimizing for. In this case, there is only one, q. We list these variables under our optimization condition.

$$\max_{q}$$

Unconstrained Optimization

- Lastly, we list our objective function. This is the goal of the problem. In this case, the monopolist is maximizing his profits,

$$\max_{q} \ \pi$$

- We can expand on this, however, since $\pi = TR - TC$. Substituting,

$$\max_{q} \ TR - TC$$

- Once more, we can substitute $TR = pq$ (and use the inverse demand function for p) and $TC = 50q$ (since the marginal cost of production is constant) to further expand on this, obtaining,

$$\max_{q} \ (250 - 2q)q - 50q$$

Unconstrained Optimization

$$\max_{q} \ (250 - 2q)q - 50q$$

- Once here, we are ready to calculate our first-order (and second-order, if need be) conditions.
- Being able to set up a problem this way will keep you organized and reduce the likelihood of making mistakes going forward.
  - As a tip, the first thing you should do when facing an optimization problem is to clearly define the problem.
    - Make sure you consider all the variables of interest as well as properly defining the objective function.

Unconstrained Optimization

$$\max_{q} \ (250 - 2q)q - 50q$$

- When we calculate first-order conditions, we take the derivative of the objective function for each of our choice variables, then set it equal to zero (since that is where a maximum or minimum occurs).
  - Thus, we calculate the derivative of our profit function with respect to q, since that is our only choice variable, then we set it equal to zero.

$$\frac{d\pi}{dq} = 250 - 4q - 50 = 0$$

  - Taking a closer look at this, notice that it's the same condition as marginal revenue equals marginal cost.

$$\underbrace{250 - 4q}_{MR} - \underbrace{50}_{MC} = 0$$

Unconstrained Optimization

$$\frac{d\pi}{dq} = 250 - 4q - 50 = 0$$

- From here, we simply solve this equation for q to find our equilibrium.
  - Rearranging terms,

$$4q = 200$$

- Lastly, dividing both sides by 4, we obtain our equilibrium quantity,

$$q^* = 50$$

- If we wanted to, we could go back and calculate our equilibrium price and profit level from this information.
  - Let's make sure we have a maximum however, first.

Unconstrained Optimization

$$\frac{d\pi}{dq} = 250 - 4q - 50 = 0$$

- Recall that for this value of q to be a maximum (rather than a minimum), the second derivative must be negative.
  - We can calculate the second derivative by just differentiating this function one more time,

$$\frac{d^2\pi}{dq^2} = -4 < 0$$

  - Since this function is negative for every possible value of q, we know that we have a maximum.

Unconstrained Optimization

$$q^* = 50$$

- Now we can calculate our equilibrium price and profit level.
  - We find our equilibrium price by plugging the equilibrium quantity back into the inverse demand function,

$$p^* = 250 - 2q^* = 250 - 2(50) = 150$$

  - Lastly, we obtain our profit level by plugging the equilibrium quantity back into the profit function,

$$\pi^* = (250 - 2q^*)q^* - 50q^* = 5{,}000$$