Inverting a Matrix

Basic Math for Economics – Refresher

Introduction

- With an understanding of basic matrix algebra and determinants, we are ready to discuss how to invert a matrix.
- Recall that matrix inversion is a useful technique that allows us to solve systems of equations directly.
  - This technique is commonly used when calculating regressions in econometrics.
- Inverting a matrix involves four steps:
  - Calculate minors.
  - Reflect across the trace.
  - Adjust signs.
  - Divide by the determinant.

Matrix Inversion

- Suppose we wanted to invert the following matrix,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- Let's follow our steps.
  - First, we need to calculate the minors.
    - In words, we need to calculate the determinant for the submatrix that encompasses all elements that are not in the same row or column as our current element.
      - For a 2x2 matrix, this is actually quite simple.

Matrix Inversion

- Suppose we wanted to calculate the minor for our first element of the inverted matrix.
  - We need to look at the determinant for every element that is not in the first row or the first column,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

  - This leaves us with just d, and the determinant of a single element in a matrix is just that element, so the element in the first row and first column of our inverted matrix is just d.

$$\begin{bmatrix} d & ? \\ ? & ? \end{bmatrix}$$

Matrix Inversion

- We can continue this process for the other elements to obtain

$$\begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

- Now we need to reflect across the trace.
  - (recall that the trace is just all of the diagonal elements of a matrix)
  - Effectively, we need to swap the element in row 1 and column 2 with the element in row 2 and column 1, etc.
    - This updates our inverted matrix to:

$$\begin{bmatrix} d & b \\ c & a \end{bmatrix}$$

Matrix Inversion

- Adjust signs.
  - We need to multiply each term of the matrix by -1^{r+c} where r is the row of the element and c is the column of the element.
    - Note that this coefficient can only take two values: 1 and -1, depending on whether $r+c$ is even or odd.
    - The element in the first row and the first column of the matrix is multiplied by $-1^{1+1} = -1^2 = 1$, and so on.
  - Updating our inverted matrix:

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Matrix Inversion

- Divide by the determinant.
  - Lastly, we divide by the determinant of the original matrix. Recall that the original matrix is,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and the formula for the determinant is,

$$|A| = ad - bc$$

  - Since this is a scalar, we can just divide each individual term in the matrix by this number, yielding our final inverted matrix of

$$A^{-1} = \begin{bmatrix} \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix}$$

Matrix Inversion

- Let's do this again, but with a 3x3 matrix.
  - Consider the following matrix,

$$B = \begin{bmatrix} 2 & 6 & 4 \\ 1 & 8 & 3 \\ 5 & 7 & 9 \end{bmatrix}$$

  - Again we just need to follow the four steps to calculate our inverse matrix:
    - Calculate minors.
    - Reflect across the trace.
    - Adjust signs.
    - Divide by the determinant.

Matrix Inversion

- Calculate minors.
  - This is a bit more complicated for our 3x3 matrix. Let's look at the minor for the element in the first row, second column (6).
    - Recall that we need to calculate the determinant of the submatrix that contains every element neither in this row nor column, i.e.,

$$B = \begin{bmatrix} 2 & 6 & 4 \\ 1 & 8 & 3 \\ 5 & 7 & 9 \end{bmatrix}$$

    - This leaves us with a 2x2 matrix,

$$\begin{bmatrix} 1 & 3 \\ 5 & 9 \end{bmatrix}$$

whose determinant is $1 * 9 - 3 * 5 = -6$

Matrix Inversion

- Thus, the minor for the first row and second column of our inverted matrix is -6,

$$\begin{bmatrix} ? & -6 & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

- Calculating the rest of our minors,

$$\begin{bmatrix} 51 & -6 & -33 \\ 26 & -2 & -16 \\ -14 & 2 & 10 \end{bmatrix}$$

  - We are now ready to move on to our next step.

Matrix Inversion

$$\begin{bmatrix} 51 & -6 & -33 \\ 26 & -2 & -16 \\ -14 & 2 & 10 \end{bmatrix}$$

- Reflect across the trace.
  - This is the same as in our 2x2 example; we just mirror the elements of the matrix across its diagonal.
    - Updating our inverse matrix,

$$\begin{bmatrix} 51 & 26 & -14 \\ -6 & -2 & 2 \\ -33 & -16 & 10 \end{bmatrix}$$

Matrix Inversion

$$\begin{bmatrix} 51 & 26 & -14 \\ -6 & -2 & 2 \\ -33 & -16 & 10 \end{bmatrix}$$

- Adjust signs.
  - Once again, we multiply each term by -1^{r+c} to adjust our signs as need be.
    - Updating our inverse matrix,

$$\begin{bmatrix} 51 & -26 & -14 \\ 6 & -2 & -2 \\ -33 & 16 & 10 \end{bmatrix}$$

Matrix Inversion

- Divide by the determinant.
  - Lastly, we divide by the determinant, which we calculate from the original matrix,

$$B = \begin{bmatrix} 2 & 6 & 4 \\ 1 & 8 & 3 \\ 5 & 7 & 9 \end{bmatrix}$$

  - Recall the formula for the determinant of a 3x3 matrix,

$$|B| = 2(8)(9) + 6(3)(5) + 4(1)(7) - 2(3)(7) - 6(1)(9) - 4(8)(5) = 6$$

  - Thus, we divide each term in our matrix by 6 to obtain our inverted matrix.

Matrix Inversion

$$\begin{bmatrix} 51 & -26 & -14 \\ 6 & -2 & -2 \\ -33 & 16 & 10 \end{bmatrix}$$

- Dividing each term by 6, we have our inverted matrix,

$$B^{-1} = \begin{bmatrix} 8.5 & -4.33 & -2.33 \\ 1 & -0.33 & -0.33 \\ -5.5 & 2.67 & 1.67 \end{bmatrix}$$

- We can use this technique for a matrix of any size.
  - Note that as matrices become larger, the calculations involved become more complicated.