Reduced Row Echelon Form

Basic Math for Economics – Refresher

Introduction

- With a basic introduction to matrix algebra, let's use a matrix to solve a system of three equations and three unknowns.
  - While this method may seem a bit tedious, it is quite powerful for organizing many different parameters.
    - As the number of equations and unknowns increases, the ease of a matrix relative to standard algebra increases dramatically.
    - This is especially important when dealing with econometric equations.

Reduced Row Echelon Form

- Suppose we were in a situation where three firms competed in quantities in a Cournot setting. You have calculated first-order conditions for each firm as follows,

$$90 - 2q_1 - q_2 - q_3 = 0$$
$$90 - q_1 - 2q_2 - q_3 = 0$$
$$90 - q_1 - q_2 - 2q_3 = 0$$

  where q_1, q_2, and q_3 represent the quantities produced by firms 1, 2 and 3, respectively.

  - We could solve this using substitution, but it would be cumbersome. Let's use a matrix.

Reduced Row Echelon Form

$$90 - 2q_1 - q_2 - q_3 = 0$$
$$90 - q_1 - 2q_2 - q_3 = 0$$
$$90 - q_1 - q_2 - 2q_3 = 0$$

- The first thing we need to do is rearrange these equations such that all of the variables (q_1, q_2, and q_3) are on the left-hand side, and everything else is on the right-hand side.

$$2q_1 + q_2 + q_3 = 90$$
$$q_1 + 2q_2 + q_3 = 90$$
$$q_1 + q_2 + 2q_3 = 90$$

Reduced Row Echelon Form

$$2q_1 + q_2 + q_3 = 90$$
$$q_1 + 2q_2 + q_3 = 90$$
$$q_1 + q_2 + 2q_3 = 90$$

- Now we form this into a matrix by using the coefficient of each variable as an element in the matrix.
  - Be sure to align the variables such that they are in the same column!

$$\begin{bmatrix} 2 & 1 & 1 & 90 \\ 1 & 2 & 1 & 90 \\ 1 & 1 & 2 & 90 \end{bmatrix}$$

Reduced Row Echelon Form

$$\begin{bmatrix} 2 & 1 & 1 & 90 \\ 1 & 2 & 1 & 90 \\ 1 & 1 & 2 & 90 \end{bmatrix}$$

- Now we are ready to convert this matrix into reduced row echelon form (to solve it).
  - We want to manipulate this matrix until all of the diagonal elements (same row and column) are equal to 1, and then everything that is not in the final column is equal to zero.
    - But how do we manipulate a matrix?

Reduced Row Echelon Form

$$\begin{bmatrix} 2 & 1 & 1 & 90 \\ 1 & 2 & 1 & 90 \\ 1 & 1 & 2 & 90 \end{bmatrix}$$

- Manipulating a matrix is relatively straightforward.
  - We can perform any operation on any row of the matrix as long as we do it to each element.
    - We can even add and subtracts rows together!
  - For example, we need the 2 in the first to become a 1 in order to achieve our reduced row echelon form.
    - So let's divide it by 2. Just remember to perform the same operation on all elements of the row.

Reduced Row Echelon Form

$$\begin{bmatrix} 2 & 1 & 1 & 90 \\ 1 & 2 & 1 & 90 \\ 1 & 1 & 2 & 90 \end{bmatrix}$$

- Dividing the first row by 2, we obtain,

$$\begin{bmatrix} 1 & 0.5 & 0.5 & 45 \\ 1 & 2 & 1 & 90 \\ 1 & 1 & 2 & 90 \end{bmatrix}$$

- Now, I need the number in the second row and first column to equal zero.
  - With that 1 I just obtained in the first row and first column, I could achieve my zero in the second row and first column by subtracting row 1 from row 2.

Reduced Row Echelon Form

$$\begin{bmatrix} 1 & 0.5 & 0.5 & 45 \\ 1 & 2 & 1 & 90 \\ 1 & 1 & 2 & 90 \end{bmatrix}$$

- Subtracting row 1 from row 2 gives us,

$$\begin{bmatrix} 1 & 0.5 & 0.5 & 45 \\ 0 & 1.5 & 0.5 & 45 \\ 1 & 1 & 2 & 90 \end{bmatrix}$$

- We can do the same thing for row 3. Subtracting row 1 from row 3 gives us,

$$\begin{bmatrix} 1 & 0.5 & 0.5 & 45 \\ 0 & 1.5 & 0.5 & 45 \\ 0 & 0.5 & 1.5 & 45 \end{bmatrix}$$

Reduced Row Echelon Form

$$\begin{bmatrix} 1 & 0.5 & 0.5 & 45 \\ 0 & 1.5 & 0.5 & 45 \\ 0 & 0.5 & 1.5 & 45 \end{bmatrix}$$

- This is a nice situation to be in.
  - Since all of the other elements in column 1 aside from the first row's is equal to 0, performing any other addition or subtraction on row 1 will no longer change the value of that first element.
- Now we move on to row 2.
  - The first thing we want to do is get the element on the diagonal equal to 1.
    - We can achieve this by dividing each element by 1.5.

Reduced Row Echelon Form

$$\begin{bmatrix} 1 & 0.5 & 0.5 & 45 \\ 0 & 1.5 & 0.5 & 45 \\ 0 & 0.5 & 1.5 & 45 \end{bmatrix}$$

- Dividing row 2 by 1.5 gives us,

$$\begin{bmatrix} 1 & 0.5 & 0.5 & 45 \\ 0 & 1 & 0.33 & 30 \\ 0 & 0.5 & 1.5 & 45 \end{bmatrix}$$

- From here, we want all of the other elements in column 2 to become zero, which we can achieve through some addition and (scalar) multiplication.
  - We just can’t add them directly this time.

Reduced Row Echelon Form

$$\begin{bmatrix} 1 & 0.5 & 0.5 & 45 \\ 0 & 1 & 0.33 & 30 \\ 0 & 0.5 & 1.5 & 45 \end{bmatrix}$$

- Let's subtract half of row 2 from row 1. This gives us,

$$\begin{bmatrix} 1 & 0 & 0.33 & 30 \\ 0 & 1 & 0.33 & 30 \\ 0 & 0.5 & 1.5 & 45 \end{bmatrix}$$

- Likewise, we'll subtract half of row 2 from row 3, obtaining,

$$\begin{bmatrix} 1 & 0 & 0.33 & 30 \\ 0 & 1 & 0.33 & 30 \\ 0 & 0 & 1.33 & 30 \end{bmatrix}$$

Reduced Row Echelon Form

$$\begin{bmatrix} 1 & 0 & 0.33 & 30 \\ 0 & 1 & 0.33 & 30 \\ 0 & 0 & 1.33 & 30 \end{bmatrix}$$

- Now we move on to the third column and do the same thing.
  - First we divide the third row by 1.33, obtaining

$$\begin{bmatrix} 1 & 0 & 0.33 & 30 \\ 0 & 1 & 0.33 & 30 \\ 0 & 0 & 1 & 22.5 \end{bmatrix}$$

    - All that is left now to do is subtract a multiplied row 3 from rows 1 and 2.

Reduced Row Echelon Form

$$\begin{bmatrix} 1 & 0 & 0.33 & 30 \\ 0 & 1 & 0.33 & 30 \\ 0 & 0 & 1 & 22.5 \end{bmatrix}$$

- Subtracting one-third of row 3 from row 1 give us,

$$\begin{bmatrix} 1 & 0 & 0 & 22.5 \\ 0 & 1 & 0.33 & 30 \\ 0 & 0 & 1 & 22.5 \end{bmatrix}$$

- Likewise, subtracting one-third of row 3 from row 2 yields,

$$\begin{bmatrix} 1 & 0 & 0 & 22.5 \\ 0 & 1 & 0 & 22.5 \\ 0 & 0 & 1 & 22.5 \end{bmatrix}$$

Reduced Row Echelon Form

$$\begin{bmatrix} 1 & 0 & 0 & 22.5 \\ 0 & 1 & 0 & 22.5 \\ 0 & 0 & 1 & 22.5 \end{bmatrix}$$

- This matrix is now in reduced row echelon form.
  - We've also solved for our three equations and three unknowns!
  - Recall that each column corresponded with one of the variables. From the first row, we have that $q_1^* = 22.5$. From the second and third rows, we have that $q_2^* = 22.5$ and $q_3^* = 22.5$.
    - We can use this technique for any number of equations and unknowns.