What is an Integral?

Basic Math for Economics – Refresher

Introduction

- With derivatives out of the way, we now need to discuss its dichotomy, the integral.
  - The integral, or antiderivative, is a method for moving from the rate of change of a variable to the total amount of change.
- Integrals come in two types: indefinite and definite.
  - Indefinite integrals focus on the pure conversion from a marginal variable to a total variable.
    - They typically do not come with an initial condition, however, which prevents a precise analysis.
      - Intuitively, we lose a constant when to calculate a derivative, and the integral does not restore that constant.

Integrals

- A definite integral specifies a range for the integral, which removes the problem of an unknown constant.
  - With a specified range, the constant cancels out, regardless of its value.
- Definite integrals are used frequently in economics.
  - Common applications are in welfare analysis and expected value calculations.
    - Basically, any situation where someone is interested in the area underneath a curve warrants the use of a definite integral.

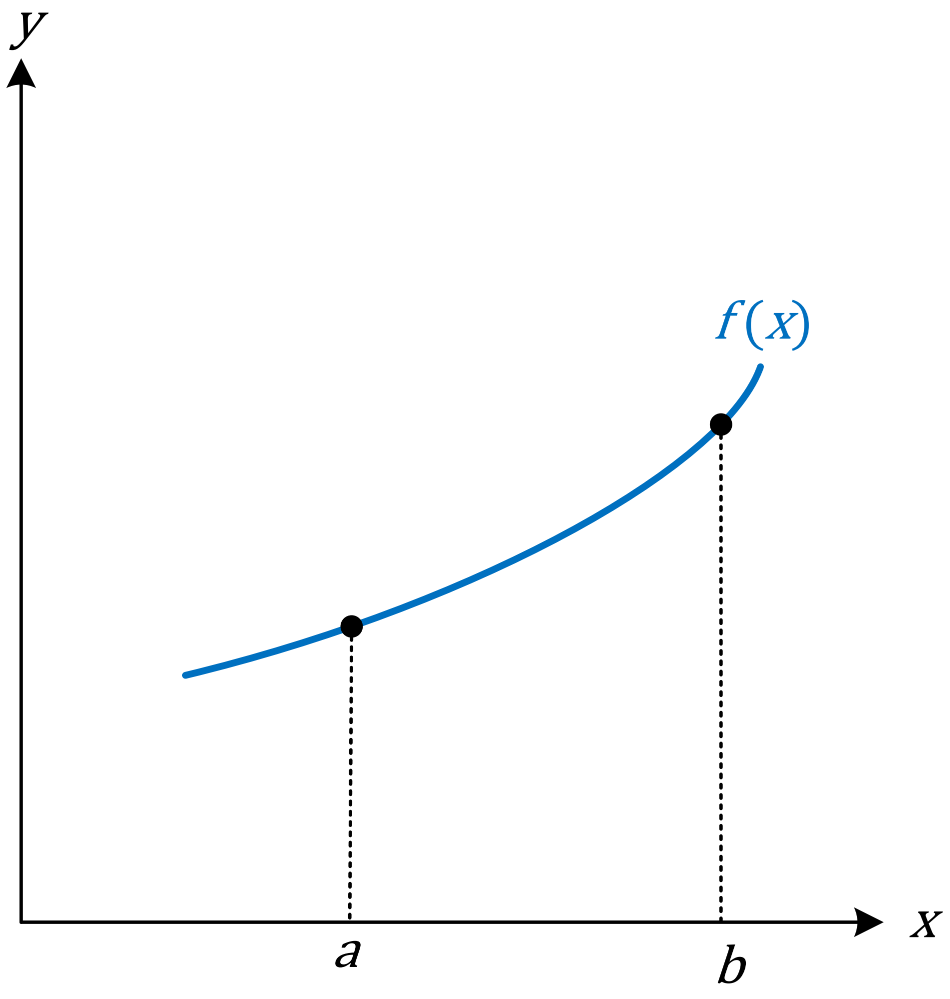
Integrals

- Formally, we can define $F(x)$ as the definite integral of the function $f(x)$ from an initial point a to a final point b as

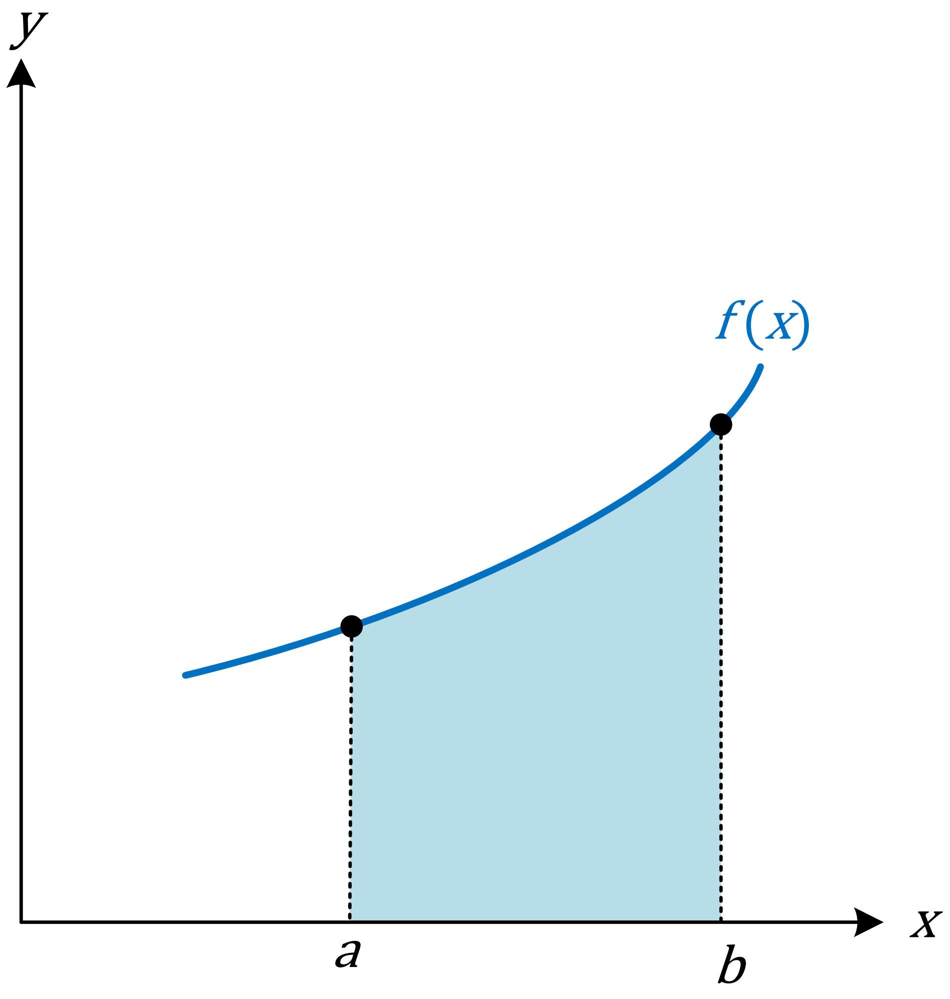
$$F(x) = \int_a^b f(x)dx$$

- Mathematically, we calculate integrals using the opposite steps that were used when calculating derivatives.
  - This allows us to reuse concepts like the power rule, logarithms, and the exponential functions.
  - In addition, we can also use techniques like the product rule and chain rule to simplify our analysis.
    - They're renamed integration by parts, and U-substitutions, respectively, but follow the same mechanics.

Integrals



Integrals



Integrals

- Let's practice a few of these.
  - Calculate the area underneath the curve $f(x) = x^2$ from 1 to 3, $\int_1^3 x^2 dx$.
    - To start, we need to calculate the integral of x^2, which we can do with the power rule.
    - Remember though that we need to work backwards, which includes the steps of the power rule.
      - First we increase the exponent of our integrand (x) by 1, then we divide the coefficient by our exponent.
      - Thus, increasing the exponent by 1 yields $2 + 1 = 3$, and dividing our coefficient (1) by the exponent (3) yield our integral, $F(x) = \frac{1}{3}x^3$

Integrals

$$F(x) = \frac{1}{3}x^3$$

- Next, we simply plug in our two endpoints, subtracting our initial point from our final point. In this case,

$$F(1) = \frac{1}{3}(1)^3 = \frac{1}{3}$$

$$F(3) = \frac{1}{3}(3)^3 = 9$$

- Thus, our area underneath the curve in this case is

$$F(3) - F(1) = 9 - \frac{1}{3} = \frac{26}{3} \approx 8.67$$

Integrals

- Let's try a multivariable integral.
  - Calculate the integral of $\frac{y}{x}$ with respect to x from 2 to 10, and with respect to y from 3 to 4, $\int_3^4 \int_2^{10} \frac{y}{x}\, dx\, dy$.
  - This one might seem a bit harder, but we just need to follow the same steps as before.
    - Let's focus on the inner integral first, $\int_2^{10} \frac{y}{x}\, dx$.
    - Like before with multivariable derivatives, we treat the y in this case as a constant and leave it alone.
      - In fact, we can even move it outside of the integral, which is appropriate for any constant, giving us $y \int_2^{10} \frac{1}{x}\, dx$.

Integrals

$$y \int_{2}^{10} \frac{1}{x} dx$$

- The inner term of this integral is now $\frac{dx}{x}$, which if we recall is the derivative of the natural logarithm, $\log(x)$. Thus, the integral is just that.
  - All that remains to calculate is the value of $\log(x)$ at both 10 and 2, then subtract the initial point from the final point.

$$\log 2 \approx 0.7$$

$$\log 10 \approx 2.3$$

- Thus, this inner term is simply $y(2.3 - 0.7) = 1.6y$.
  - Don’t forget about the y term!

Integrals

- Updating our integral,

$$\int_3^4 \int_2^{10} \frac{y}{x} dx\, dy = \int_3^4 1.6y dy$$

  - Notice that after integrating with respect to x, all instances of the variable x are removed from the integral, but the variable y remains unchanged.
  - Now, let's calculate our outer integral.
    - Integrating $1.6y$ with respect to y is simply an application of the power rule.
      - Adding one to the exponent, then dividing by that exponent gives us $0.8y^2$

Integrals

$$\int_{3}^{4} 1.6y dy = 0.8y^2 \Big|_3^4$$

- From here, we simply plug in our two endpoints for y and subtract the initial endpoint from the final endpoint.

$$0.8(3)^2 = 7.2$$

$$0.8(4)^2 = 13.28$$

- This leads us to our total area underneath the curve of $13.28 - 7.2 = 6.08$.
  - As before, we just follow the steps, and these calculations are relatively straightforward.