Multivariable Derivatives

Basic Math for Economics - Refresher

Introduction

- Most of the time in economics, we deal with optimization problems that contain more than one variable.
  - Consumer choice, Cournot competition, etc.
- In these cases, we must make use of multivariable calculus to optimize these problems; namely, partial derivatives.
  - The good news is that these are not that much more complicated that their single variable counterparts.
    - The trick is to treat everything that is not being differentiated as a constant.

Multivariable Derivatives

- Let's look at two examples for this topic: the first from consumer theory.
- Suppose a consumer could purchase two goods: x and y, and receives the following utility from their consumption:

$$U(x, y) = x^{0.4}y^{0.6}$$

- Let's calculate their marginal rate of substitution,

$$MRS = -\frac{MU_x}{MU_y}$$

where the numerator and denominator represent the marginal utilities for goods x and y respective, or the partial derivatives of the utility function for each variable.

Multivariable Derivatives

$$U(x, y) = x^{0.4}y^{0.6}$$

- Starting with good x, let's calculate a partial derivative.
  - Since we are dealing with good x, we treat good y as if it were constant.
    - Intuitively, we are looking at only that rate of change of x, so holding y constant makes sense.
  - Going back to our power rule, we first multiply the coefficient by the exponent, then decrease the exponent by 1, which gives us

$$MU_x = U_x(x, y) = 0.4x^{-0.6}y^{0.6}$$

    - Notice that the y term is completely left alone. This is because it is a part of the same term as x.

Multivariable Derivatives

$$U(x, y) = x^{0.4}y^{0.6}$$

- Next, we do the same thing for good y, applying the power rule again yields,

$$MU_y = U_y(x, y) = 0.6x^{0.4}y^{-0.4}$$

  - Once again, the x term is treated as a constant and left alone.

- Lastly, we can assemble these two expressions into the marginal rate of substitution,

$$MRS = -\frac{MU_x}{MU_y} = -\frac{0.4x^{-0.6}y^{0.6}}{0.6x^{0.4}y^{-0.4}}$$

$$MRS = -\frac{2y}{3x}$$

Multivariable Derivatives

- Now let's look at an example of Cournot competition.
- Suppose two firms competed as a duopoly in a market and faced the inverse market demand curve of

$$p = 100 - q_1 - q_2$$

  where q_1 and q_2 are firms 1 and 2's quantities, respectively.
- Let's calculate the marginal revenue for firm 1.
  - Similar to before, we calculate the marginal revenue by calculating a partial derivative of the total revenue function with respect to firm 1's quantity.

Multivariable Derivatives

$$p = 100 - q_1 - q_2$$

- Recall that the total revenue function is just the price times the quantity, which we can substitute in the inverse demand function to obtain,

$$TR_1 = p \cdot q_1 = (100 - q_1 - q_2)q_1$$

- Using the product rule, we have $g(q_1, q_2) = 100 - q_1 - q_2$ and $h(q_1, q_2) = q_1$.
  - We can use the power rule on each of them to calculate the partial derivatives with respect to q_1, $g_{q_1}(q_1, q_2) = -1$ and $h_{q_1}(q_1, q_2) = 1$.
    - Note that the partial derivative of $g(q_1, q_2)$ does not leave q_2 alone. Since it is not multiplied by q_1, it's just a constant and the derivative of a constant is zero!

Multivariable Derivatives

- Using our product rule definition,

$$MR_1 = f_{q_1}(q_1, q_2) = g_{q_1}(q_1, q_2) \cdot h(q_1, q_2) + g(q_1, q_2) \cdot h_{q_1}(q_1, q_2)$$

$$MR_1 = -1 \cdot q_1 + (100 - q_1 - q_2) \cdot 1 = 100 - 2q_1 - q_2$$

- We can perform similar calculations for firm 2 and obtain,

$$MR_2 = 100 - q_1 - 2q_2$$

- The general idea of a partial derivative is to treat all variables except the one that we are differentiating with respect to as constant.
  - If the variables are in the same term, leave them alone. Otherwise, recall that the derivative of a constant is zero.