Product and Chain Rules

Basic Math for Economics – Refresher

Introduction

- There are times when calculating the derivative of a function is too cumbersome to simply use the power rule.
  - It could be a situation where one function is multiplied by another, or it could be a function within a function.
    - While it may be possible to distribute everything and then apply the power rule, it might be much easier to use a different technique.
    - We classify these techniques into two main types (there are others that are rarely used in economics), depending on how the functions are organized: the product rule and the chain rule.

Product Rule

- The product rule is useful when we have situations where two functions are multiplied by each other.
  - Suppose we have functions $g(x)$ and $h(x)$. We can define function $f(x)$ as,

$$f(x) = g(x) \cdot h(x)$$

- The Product Rule gives us a quick way to calculate the derivative of $f(x)$ and is defined as follows:

$$f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x)$$

  - In words, the derivative of $f(x)$ is simply the derivative of the first function times the second function plus the derivative of the second function times the first function.

Product Rule

- Let's practice this by calculating the marginal revenue function.
- Suppose a monopolist faced the inverse demand function

$$p = 100 - 2q$$

  - We can find the monopolist's total revenue function by multiplying the price they charge and the quantity they sell, $TR = p \cdot q$.
    - In fact, we can make this solely a function of the quantity by substituting the inverse demand function for p,

$$TR = p \cdot q = (100 - 2q) \cdot q$$

Product Rule

$$TR = (100 - 2q)q$$

- We could easily solve this by distributing the q through the parenthesis, but let's apply the product rule.
  - We have two functions multiplied by each other, $g(q) = 100 - 2q$ and $h(q) = q$.
  - From the power rule, we can obtain the derivatives of each of these functions, $g'(q) = -2$ and $h'(q) = 1$.
  - Using our definition of the product rule,

$$\text{MR} = f'(q) = g'(q) \cdot h(q) + g(q) \cdot h'(q) = -2 \cdot q + (100 - 2q) \cdot 1$$

$$MR = -2q + 100 - 2q = 100 - 4q$$

Chain Rule

- Lastly, we have the chain rule. The chain rule is useful when we are dealing with functions of functions.
  - Suppose we have functions $g(x)$ and $h(x)$. We can define function $f(x)$ as,

$$f(x) = g\big(h(x)\big)$$

- The chain rule gives us a quick way to calculate the derivative of $f(x)$ and is defined as follows:

$$f'(x) = g'\big(h(x)\big) \cdot h'(x)$$

  - In words, the derivative of $f(x)$ is simply the derivative of the outer function (where the inner function acts as the argument of the outer function) times the derivative of the inner function.

Chain Rule

- Let's look at the chain rule through another utility function, $U(x) = \log(x^2)$. Let's calculate the marginal utility, $U'(x)$.
  - In this case, $g(x) = \log x$ and $h(x) = x^2$. In this case, $h(x)$ serves as the argument of $g(x)$, but is also a function of x.
  - As before, we can calculate both of their derivatives as $g'(x) = \frac{1}{x}$ and $h'(x) = 2x$.
- Using our definition of the chain rule,

$$U'(x) = g'\big(h(x)\big) \cdot h'(x) = \frac{1}{h(x)} \cdot 2x$$

$$U'(x) = \frac{2x}{x^2} = \frac{2}{x}$$