Commonly Used Derivatives

Basic Math for Economics – Refresher
Let’s continue our discussion of derivatives by looking at the three most common types we see in economics.

- **The Power Rule** – The overwhelming majority of derivatives done in economics can be done as an application of this rule.

- **Natural Logarithms** – This is a common functional form used for utility functions.

- **Exponential Function** – This is a common functional form for demand functions.

Each of these types of functions have their own techniques for calculating their derivatives.
Power Rule

- The Power Rule can be used on almost any polynomial to calculate its derivative.
  - (a polynomial is just a series of exponents of a variable added together, like $x^4 + 2x + 4$.)
- Let’s look at the Power Rule through an example. Consider the following total cost function:
  $$T(q) = 10 + 3q + 4q^2 + q^3$$
- Let’s take this function and calculate the marginal cost function.
  - Recall that the marginal cost function is just the derivative of the total cost function.
$T(q) = 10 + 3q + 4q^2 + q^3$

- In order to calculate the derivative of this function using the Power Rule, we follow a few steps:
  - First, if a term does not contain the variable we’re differentiating with respect to ($q$ in this case), we delete it.
  - Second, if a term does contain the variable we’re differentiating with respect to, we multiply that variable’s coefficient by its exponent, then decrease that variable’s exponent by one.

- Let’s walk through this one.
Starting with the first term of this expression \((10)\), it does not contain our variable, \(q\).

Thus, we simply delete it.

Now, we move on to our second term, \(3q\). Since it has the variable \(q\), we apply the Power Rule.

First, we multiply the coefficient of the variable (3), by the variable’s exponent (1), and the coefficient for this term in our marginal cost is \(3 \times 1 = 3\).

Next, we reduce the exponent for \(q\) by one. This gives us \(q^{1-1} = q^0 = 1\).

Thus, the first term our marginal cost function is simply 3.
Moving on to the next term ($4q^2$), we once again apply the Power Rule since it contains the variable $q$.

First, we multiply this term’s coefficient (4) by its exponent (2), and obtain the coefficient for this term in our derivative, $4 \times 2 = 8$.

Next, we decrease the exponent of the variable by one, obtaining $q^{2-1} = q^1 = q$.

Thus, the second term of our marginal cost function is $8q$. 

\[ T(q) = 10 + 3q + 4q^2 + q^3 \]
$T(q) = 10 + 3q + 4q^2 + q^3$

- Lastly, we examine the final term of our total cost function ($q^3$). Once again, it contains the variable $q$, so we apply the power rule.

  - First, we multiply the coefficient of the variable (1) by its exponent (3), and we obtain the coefficient for this term in our derivative, $3 \times 1 = 3$.

  - Next, we decrease the exponent of our variable by one, obtaining $q^{3-1} = q^2$.

    - Thus, the last term of our marginal cost function is $3q^2$. 

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Power Rule

\[ T(q) = 10 + 3q + 4q^2 + q^3 \]

- Putting all of our terms together, we have our marginal cost function,
  \[ MC(q) = 3 + 8q + 3q^2 \]

- This technique is general in the sense that it will work on almost any polynomial.
  - The exponents don’t even have to be integers, either, which is common with utility and production functions.
    - Just follow the same steps.
  - Some functions, however, are a bit more complicated, and we have rules to simplify their analysis.
    - Those rules build on the power rule, and we will discuss them in the next session.
Natural Logarithm

- The Natural Logarithm is commonly seen in economics as a functional form for utility functions.

- We really don’t use any other bases for logarithms in economics. It’s safe to assume that if someone writes \( \log x \), they are referring to the natural logarithm, which we will do here.

- For a function of the form \( \log[g(x)] \), its derivative is simply

\[
 f'(\log[g(x)]) = \frac{g'(x)}{g(x)}
\]

- Let’s look at another example.
Natural Logarithm

- Consider the utility function
  \[ u(x) = \log(2x) \]

- Let's calculate their marginal utility with respect to \( x \).
  - Once again, the marginal utility is just the derivative of the utility function with respect to \( x \).

- Following our formula,
  \[ f'(\log[g(x)]) = \frac{g'(x)}{g(x)} \]
  - In this case, \( g(x) = 2x \), and we can use the Power Rule to calculate \( g'(x) \).
Natural Logarithm

\[ u(x) = \log(2x) \]

○ Since \(2x\) is a function of \(x\), we can apply the power rule as normal.

○ First, we multiply its coefficient (2) by its exponent (1), to obtain the coefficient of \(g'(x)\), \(2 \times 1 = 2\).

○ Second, we decrease the exponent of \(x\) by one, obtaining \(x^{1-1} = x^0 = 1\).

○ Thus, \(g'(x) = 2\) in this case.

○ Putting this together with our Natural Logarithm definition, we obtain our derivative,

\[
\frac{d}{dx} \left( \log(2x) \right) = \frac{2}{2x} = \frac{1}{x}
\]
Exponential Function

- Lastly, as the dichotomy of the Natural Logarithm, the exponential function is commonly used in demand functions.
  - Example: the demand function \( q(p) = e^{-p} \) provides us with a demand function that is similar to the shape of one that we can derive experimentally.

- For a functional form of \( e^{g(x)} \), its derivative is simply
  \[
  f'(e^{g(x)}) = g'(x)e^{g(x)}
  \]
  - Once again, let’s look at this through an example.
Consider the demand function

$$q(p) = e^{-p^2}$$

Let’s calculate the price-elasticity of demand for any given price, $p$.

Recall that the price-elasticity of demand formula is

$$\varepsilon_{q,p} = \frac{dq(p)}{dp} \frac{p}{q(p)} = f'(e^{-p^2}) \frac{p}{e^{-p^2}}$$

In order to calculate this value, we need to calculate the derivative of $e^{-p^2}$, which requires use of both the Power Rule and our derivative of the Exponential Function.
Exponential Function

\[ q(p) = e^{-p^2} \]

- From our definition of the derivative of the Exponential Function,
  \[ f'(e^{g(x)}) = g'(x)e^{g(x)} \]

- In this case, \( g(x) = -x^2 \) (I have replaced the \( p \) with an \( x \) for simplicity). We can calculate \( g'(x) \) using the Power Rule.

  - Since our variable is a function of \( x \), first we multiply its coefficient \((-1)\) by its exponent \((2)\), to obtain the coefficient of \( g'(x) \), \(-1 \times 2 = -2\).

  - Next, we decrease the exponent by one, obtaining \( x^{2-1} = x^1 = x \)

  - Thus, \( g'(x) = -2x \) in this case.
**Exponential Function**

\[ q(p) = e^{-p^2} \]

- Returning to the definition of the derivative of the Exponential Function,
  \[ f'(e^{-p^2}) = f'(e^{g(x)}) = g'(x)e^g(x) = -2xe^{-x^2} \]

- And replacing all instances of \( x \) with the original \( p \), we have
  \[ f'(e^{-p^2}) = -2pe^{-p^2} \]

- Lastly, we substitute this into our formula for the price-elasticity of demand to obtain,
  \[ \varepsilon_{q,p} = \frac{dq}{dp} \frac{p}{q(p)} = f'(e^{-p^2}) \frac{p}{e^{-p^2}} = (-2pe^{-p^2}) \frac{p}{e^{-p^2}} = -2p^2 \]