Commonly Used Derivatives

Basic Math for Economics – Refresher

Introduction

- Let’s continue our discussion of derivatives by looking at the three most common types we see in economics.
  - The Power Rule – The overwhelming majority of derivatives done in economics can be done as an application of this rule.
  - Natural Logarithms – This is a common functional form used for utility functions.
  - Exponential Function – This is a common functional form for demand functions.
    - Each of these types of functions have their own techniques for calculating their derivatives.

Power Rule

- The Power Rule can be used on almost any polynomial to calculate its derivative.
  - (a polynomial is just a series of exponents of a variable added together, like $x^4 + 2x + 4$.)
- Let's look at the Power Rule through an example. Consider the following total cost function:

$$T(q) = 10 + 3q + 4q^2 + q^3$$

- Let's take this function and calculate the marginal cost function.
  - Recall that the marginal cost function is just the derivative of the total cost function.

Power Rule

$$T(q) = 10 + 3q + 4q^2 + q^3$$

- In order to calculate the derivative of this function using the Power Rule, we follow a few steps:
  - First, if a term does not contain the variable we're differentiating with respect to (q in this case), we delete it.
  - Second, if a term does contain the variable we're differentiating with respect to, we multiply that variable's coefficient by its exponent, then decrease that variable's exponent by one.
    - Let's walk through this one.

Power Rule

$$T(q) = 10 + 3q + 4q^2 + q^3$$

- Starting with the first term of this expression (10), it does not contain our variable, q.
  - Thus, we simply delete it.
- Now, we move on to our second term, $3q$. Since it has the variable q, we apply the Power Rule.
  - First, we multiply the coefficient of the variable (3), by the variable's exponent (1), and the coefficient for this term in our marginal cost is $3 * 1 = 3$.
  - Next, we reduce the exponent for q by one. This gives us $q^{1-1} = q^0 = 1$.
    - Thus, the first term our marginal cost function is simply 3.

Power Rule

$$T(q) = 10 + 3q + 4q^2 + q^3$$

- Moving on to the next term ($4q^2$), we once again apply the Power Rule since it contains the variable q.
  - First, we multiply this term's coefficient (4) by its exponent (2), and obtain the coefficient for this term in our derivative, $4 * 2 = 8$.
  - Next, we decrease the exponent of the variable by one, obtaining $q^{2-1} = q^1 = q$.
    - Thus, the second term of our marginal cost function is $8q$.

Power Rule

$$T(q) = 10 + 3q + 4q^2 + q^3$$

- Lastly, we examine the final term of our total cost function (q^3). Once again, it contains the variable q, so we apply the power rule.
  - First, we multiply the coefficient of the variable (1) by its exponent (3), and we obtain the coefficient for this term in our derivative, $3 * 1 = 3$.
  - Next, we decrease the exponent of our variable by one, obtaining $q^{3-1} = q^2$.
    - Thus, the last term of our marginal cost function is $3q^2$.

Power Rule

$$T(q) = 10 + 3q + 4q^2 + q^3$$

- Putting all of our terms together, we have our marginal cost function,

$$MC(q) = 3 + 8q + 3q^2$$

- This technique is general in the sense that it will work on almost any polynomial.
  - The exponents don't even have to be integers, either, which is common with utility and production functions.
    - Just follow the same steps.
  - Some functions, however, are a bit more complicated, and we have rules to simplify their analysis.
    - Those rules build on the power rule, and we will discuss them in the next session.

Natural Logarithm

- The Natural Logarithm is commonly seen in economics as a functional form for utility functions.
  - We really don't use any other bases for logarithms in economics. It's safe to assume that if someone writes $\log x$, they are referring to the natural logarithm, which we will do here.
- For a function of the form $\log[g(x)]$, its derivative is simply

$$f'(\log[g(x)]) = \frac{g'(x)}{g(x)}$$

  - Let's look at another example.

Natural Logarithm

- Consider the utility function

$$u(x) = \log(2x)$$

- Let's calculate their marginal utility with respect to x.
  - Once again, the marginal utility is just the derivative of the utility function with respect to x.
- Following our formula,

$$f'(\log[g(x)]) = \frac{g'(x)}{g(x)}$$

  - In this case, $g(x) = 2x$, and we can use the Power Rule to calculate $g'(x)$.

Natural Logarithm

$$u(x) = \log(2x)$$

- Since $2x$ is a function of x, we can apply the power rule as normal.
  - First, we multiply its coefficient (2) by its exponent (1), to obtain the coefficient of $g'(x)$, $2 * 1 = 2$.
  - Second, we decrease the exponent of x by one, obtaining $x^{1-1} = x^0 = 1$.
  - Thus, $g'(x) = 2$ in this case.
- Putting this together with our Natural Logarithm definition, we obtain our derivative,

$$f'(\log(2x)) = f'(\log[g(x)]) = \frac{g'(x)}{g(x)} = \frac{2}{2x} = \frac{1}{x}$$

Exponential Function

- Lastly, as the dichotomy of the Natural Logarithm, the exponential function is commonly used in demand functions.
  - Example: the demand function $q(p) = e^{-p}$ provides us with a demand function that is similar to the shape of one that we can derive experimentally.
- For a functional form of $e^{g(x)}$, its derivative is simply

$$f'\left(e^{g(x)}\right) = g'(x)e^{g(x)}$$

  - Once again, let's look at this through an example.

Exponential Function

- Consider the demand function

$$q(p) = e^{-p^2}$$

- Let's calculate the price-elasticity of demand for any given price, p.
  - Recall that the price-elasticity of demand formula is

$$\varepsilon_{q,p} = \frac{dq(p)}{dp}\frac{p}{q(p)} = f'\left(e^{-p^2}\right)\frac{p}{e^{-p^2}}$$

    - In order to calculate this value, we need to calculate the derivative of e^{-p^2}, which requires use of both the Power Rule and our derivative of the Exponential Function.

Exponential Function

$$q(p) = e^{-p^2}$$

- From our definition of the derivative of the Exponential Function,

$$f'\left(e^{g(x)}\right) = g'(x)e^{g(x)}$$

- In this case, $g(x) = -x^2$ (I have replaced the p with an x for simplicity). We can calculate $g'(x)$ using the Power Rule.
  - Since our variable is a function of x, first we multiply its coefficient (-1) by its exponent (2), to obtain the coefficient of $g'(x)$, $-1 * 2 = -2$.
  - Next, we decrease the exponent by one, obtaining $x^{2-1} = x^1 = x$
    - Thus, $g'(x) = -2x$ in this case.

Exponential Function

$$q(p) = e^{-p^2}$$

- Returning to the definition of the derivative of the Exponential Function,

$$f'\left(e^{-p^2}\right) = f'\left(e^{g(x)}\right) = g'(x)e^{g(x)} = -2xe^{-x^2}$$

- And replacing all instances of x with the original p, we have

$$f'\left(e^{-p^2}\right) = -2pe^{-p^2}$$

- Lastly, we substitute this into our formula for the price-elasticity of demand to obtain,

$$\varepsilon_{q,p} = \frac{dq}{dp}\frac{p}{q(p)} = f'\left(e^{-p^2}\right)\frac{p}{e^{-p^2}} = \left(-2pe^{-p^2}\right)\frac{p}{e^{-p^2}} = -2p^2$$