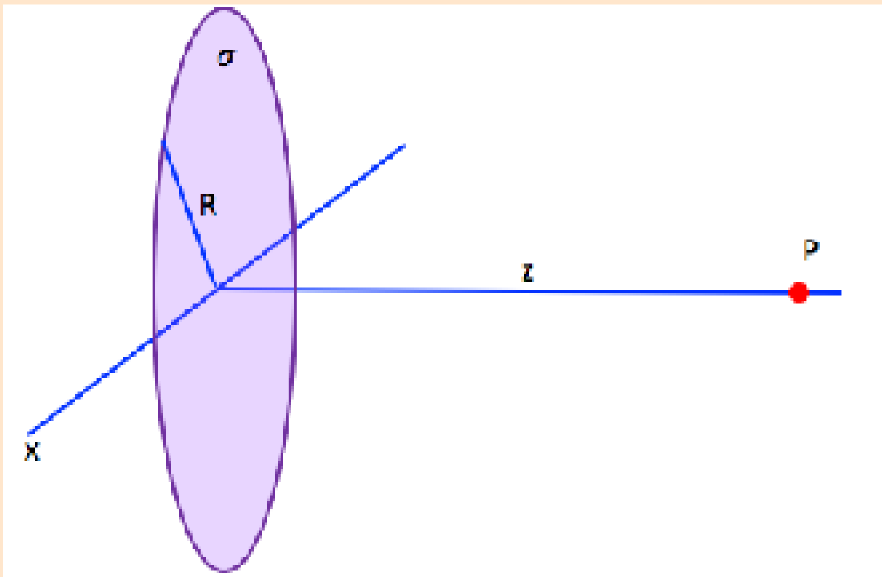
Solution to Potential and E-Field of Charged Disk ALONG Symmetry Axis

Problem: Consider a disk of radius R with a uniform charge density σ. Find the Electric Field due to this charge distribution on the axis of symmetry (z axis) for both z > 0 and z < 0. Denote the distance along the z axis from the center of the disk (O) to the point P (on the z axis) by z.



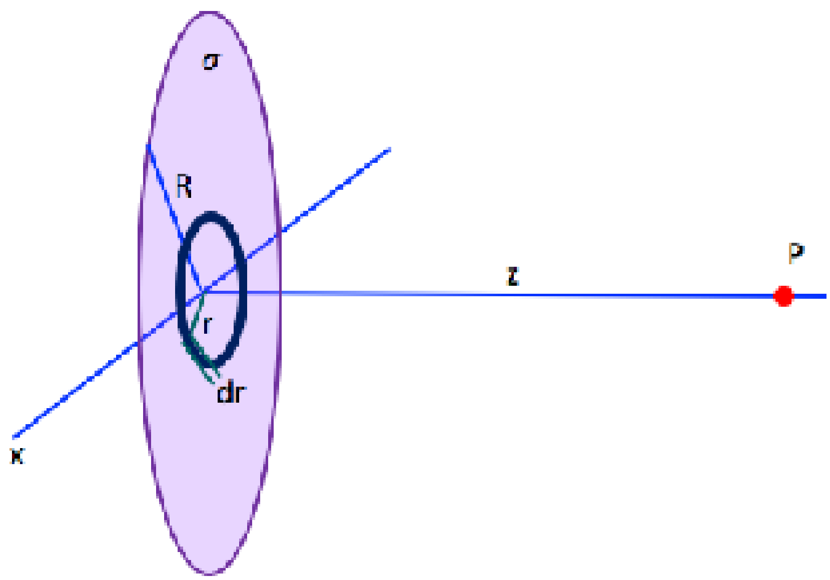
A couple of reminders:

1. (* This is a comment *)

and

2. If you get some weird results at some point, try going back and re-executing all the previous cells to reset your functions and variables.

(a) Start by finding a differential of electric potential dVring[z] (dVring as a function of z) at the point P due to a **ring of charge** of a radius between r and r + dr. Then integrate over r to find Vdisk[z].



First, the potential of a ring of charge with a uniform linear charge density and radius R was found to be equal to:

$V_{ring} = \frac{2 R k \pi \lambda}{\sqrt{R^2+z^2}}$; $\quad (k = \frac{1}{4 \pi \epsilon 0})$

But the equivalent line charge density is $\lambda = \frac{Q_{ring}}{2 \pi R}$

which ⇒ $V_{ring} = \frac{2 R k \pi \frac{Q_{ring}}{2 \pi R}}{\sqrt{R^2+z^2}} = \frac{k Q_{ring}}{\sqrt{R^2+z^2}}$.

For our ring: $Q_{ring} = Q_{ring\ of\ radius\ r\ and\ thickness\ dr} = 2 \pi r\, dr\, \sigma$; (you could call this dQring, a differential of charge, if you wished)

ding-dong: note that here we use r, not R because we are going to integrate over r.

V_{ring} **turns into a differential,** dV_{disk}**:**

The differential of the potential due to this ring of radius r $= dV_{disk}[z] = \frac{k\, 2 \pi r\, dr\, \sigma}{\sqrt{r^2+z^2}}$.

We integrate over r from 0 to R to get Vdisk[z].

(b) Enter your equation for dVring[z] in an appropriate integral written in Mathematica and thereby find the equation for Vdisk[z]. A series of assumptions (more than you need) are provided that make the integration run like a champ. They start with: $Assumptions = ...

Reminder: *To execute each cell, click your mouse anywhere inside the cell and then hit Shift-Return (Shift-Enter)*

Below is an input cell you can use for finding V[z] :

In[118]:=
```
(* Input Cell *)
ClearAll["`*"]
$Assumptions = R > 0 && R ∈ Reals &&
   z ∈ Reals && z ≠ 0 && k ∈ Reals && k > 0 && σ ∈ Reals && σ > 0;
(*  ENTER in this cell (below this comment) your dVdisk[z] into
 the appropiate integral  --reminder:  Limits needed *)
```

$$\text{Vdisk}[z\_] = \int_0^R k \frac{2\pi r \sigma}{\sqrt{r^2 + z^2}}\, dr \quad \text{(* Here is } dV_{disk}[z] \text{ inserted}$$

```
 into the integral needed to obtain Vdisk  -- note limits *)
Vdisk[
 z]
```

Out[120]= $2 k \pi \sigma \left(\sqrt{R^2 + z^2} - \text{Abs}[z]\right)$

Out[121]= $2 k \pi \sigma \left(\sqrt{R^2 + z^2} - \text{Abs}[z]\right)$

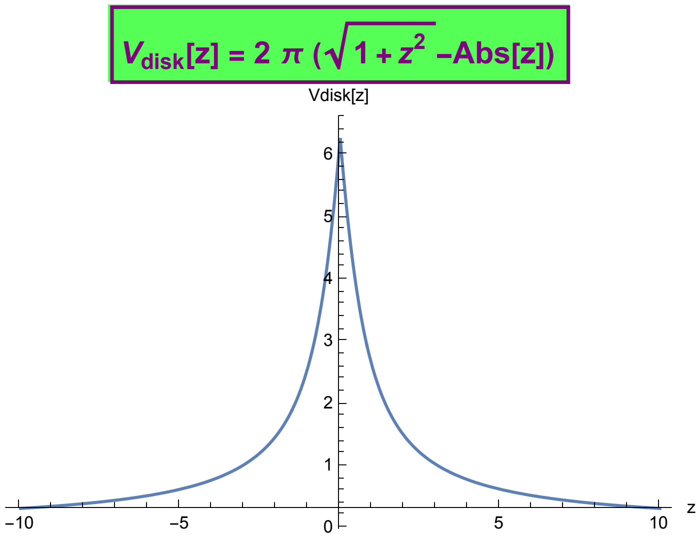
You should now have the scalar function V[z] defined as an algebraic expression which contains the parameters k, R, λ, and the variable z.

(c) Assume some values for k, R, and σ; plot Vdisk[z] from some -z $_o$ to + z $_o$.

I chose k =1, R=1, and σ=1 and I set z $_o$ = 10. [I suggest you use a σ >0 to help interpret your result.]

In[122]:=
```
(* Input Cell -- enter your code and evaluate*)
k = 1; σ = 1; R = 1;
Plot[Vdisk[z], {z, -10, 10}, PlotRange -> All, AxesLabel → {"z", "Vdisk[z]"},
 PlotLabel → Style[Framed["V_disk[z] = 2 π (√(1 + z^2) -Abs[z])"],
   16, Purple, Bold, Background → Lighter[Green]]]
```

Out[123]=



(d) Clear[k,σ, R] and then evaluate Vdisk[z] to see that it looks ok.
(I'll do it for you.)

In[124]:=
```
Clear[k, σ, R]
Vdisk[z] (*I've entered it for you; just evaluate the cell  *)
```

Out[125]= $2\,k\,\pi\,\sigma\left(\sqrt{R^2+z^2} - \text{Abs}[z]\right)$

(e) Interpret the plot of Vdisk[z] you obtained.

Enter your Discussion (this is a text cell):

Vdisk[z] should exhibit a mirror image about z = 0, along the z axis. The reason: For a positive σ, the charge on the disk is positive and therefore, because of the symmetry Vdisk[$+z_o$] = Vdisk[$-z_o$] (where $z_o > 0$). The graph shows this mirror symmetry.

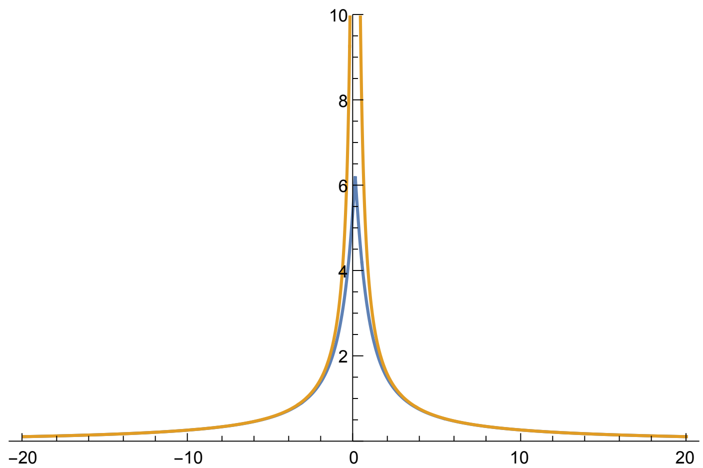
(FYI- extra info)
You should note that in both directions, the potential falls to very small values for z >> R; the disk starts looking like a point charge. The total charge on the disk is:

$$Q_{disk} = (\text{Area of Disk}) * \sigma = \pi R^2 \sigma$$

Here we plot V_{disk}[z] and the potential of a point charge: $\frac{k\,Q_{disk}}{\text{Abs[z]}}$ (We insert the Abs[z] to replace z because for $+Q_{disk}$, V of a point charge is positive for both +z and -z; Abs[z] takes care of this).

In[126]:=
```
Qdisk = π R^2 σ;
k = 1; σ = 1; R = 1;
Plot[{Vdisk[z], k Qdisk/Abs[z]}, {z, -20, 20}, PlotRange → {0, 10}]
```

Out[128]=



As expected, at sufficiently large |z|, the two potentials (the disk and the point charge) are indistinguishable; both go to zero for z -> ∞.

(f) So let's handshake on the presence of a term in Vdisk[z] containing Abs[z]. To find the E field from the potential M or we must take derivatives. M does not like taking the derivative of Abs[z] with respect to z.

We make life a lot easier for M and for ourselves if we divide the solution into two parts for z>0 and z<0. We can combine them with an If statement. It will look like this:

Vdisk[z_] = If[z < 0, (*Stick in here your Vdisk[z] for z < 0*), If[z > 0, (*Stick in here your Vdisk[z] for z > 0*)]].

M is happy taking derivatives of Vdisk[z] in the form of the If statement; it simply performs it for each part separately.

Thinking carefully, determine the appropriate Vdisk[z] for the two signs of z and enter into the input cell below your resulting Vdisk[z] [in the form of an If statement]. Execute the cell so M has Vdisk[z] defined. I stick in a Clear[k, σ, R] to keep things honest.

In[129]:=
```
(*  Input Cell  - write your Vdisk[z] for both signs of z;
you can use an If statement OR the Piecewise function  *)
Clear[k, σ, R]
Vdisk[z_] = If[z < 0, 2 k π σ (Sqrt[R^2 + z^2] + z), If[z > 0, 2 k π σ (Sqrt[R^2 + z^2] - z)]]
```

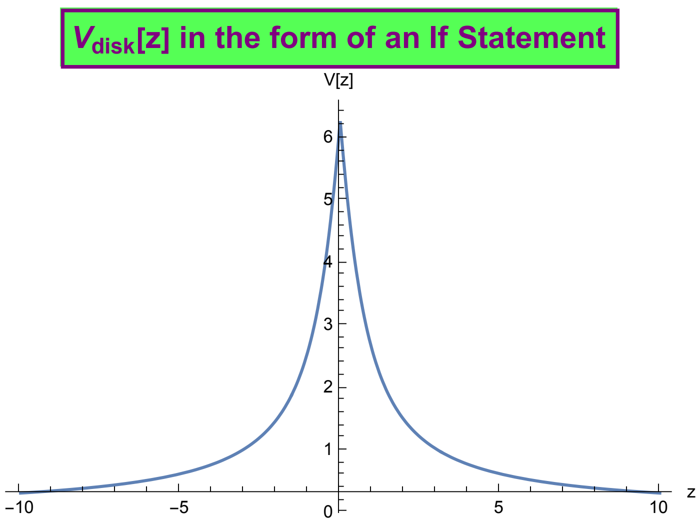
Out[130]= $\text{If}\left[z < 0,\ 2 k \pi \sigma \left(\sqrt{R^2 + z^2} + z\right),\ \text{If}\left[z > 0,\ 2 k \pi \sigma \left(\sqrt{R^2 + z^2} - z\right)\right]\right]$

(g) For comparison with the plot of Vdisk[z] above, Plot this new Vdisk[z] for the same k, σ, R and for the same z_o, over the range $-z_o$ to $+z_o$.

As before, I used k =1, σ=1, R = 1; I set z_o = 10.

In[131]:=
```
k = 1; σ = 1; R = 1;
Plot[Vdisk[z], {z, -10, 10}, PlotRange -> All, AxesLabel → {"z", "V[z]"},
 PlotLabel → Style[Framed["Vdisk[z] in the form of an If Statement"],
   16, Purple, Bold, Background → Lighter[Green]]]
```

Out[132]=



The plots for the two forms of Vdisk[z] are the same, i.e., for both ± z; the SIGN of z automatically handled this for the first version.

(h) Now find the Electric Field, E_{disk}. I used M's Grad function in Cartesian Coordinates (which generates a VECTOR).

In[133]:=
```
Clear[k, σ, R];
Vdisk[z] (* just to check to see it is still defined *)
```

Out[134]= $\text{If}\left[z < 0,\ 2 k \pi \sigma \left(\sqrt{R^2 + z^2} + z\right),\ \text{If}\left[z > 0,\ 2 k \pi \sigma \left(\sqrt{R^2 + z^2} - z\right)\right]\right]$

In[135]:=
```
Edisk[z_] = -Grad[Vdisk[z], {x, y, z}]
  (* Vdisk is a function of z only -- therefore Ediskx and Edisky = 0; note that
   the If statement form remains intact when the Grad operator is applied  *)
```

Out[135]= $\left\{0, 0, -\text{If}\left[z < 0, \left(1 + \frac{z}{\sqrt{R^2 + z^2}}\right)(2\,k\,\pi\,\sigma), \text{If}\left[z > 0, \left(-1 + \frac{z}{\sqrt{R^2 + z^2}}\right)(2\,k\,\pi\,\sigma)\right]\right]\right\}$

This generates a 3D vector (for both signs of z_o where the z component is the only non-zero component (reasonable: Vdisk[z] depends only on z; therefore, at the point P, $\overrightarrow{\text{Edisk}}$[z] = Ediskz[z] $\hat{z}$

Using Vdisk[z] for the potential we get an If statement in the resulting $\overrightarrow{\text{Edisk}}$[z]. Looking carefully, we can conclude that $\overrightarrow{\text{Edisk}}$[z] for z < 0 equals (-) $\overrightarrow{\text{Edisk}}$[z] for z > 0. [Hopefully you agree that: $\overrightarrow{\text{Edisk}}$[z] = Ediskz[z] $\hat{z}$.]

(i) Now Plot Ediskz[z] for the same k, σ, R and for the same z_o, i.e., over the same range $-z_o$ to $+z_o$ you used above. [You will need to grab the z component of $\overrightarrow{\text{Edisk}}$[z], which we call Ediskz[z].]

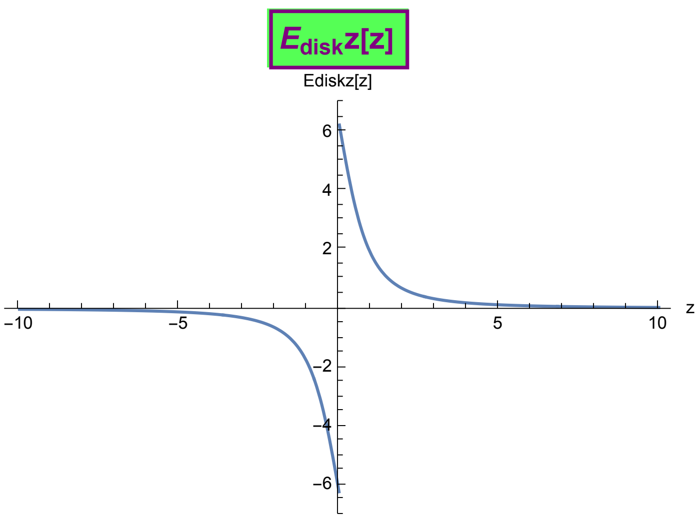
In[136]:=
```
(*  Input Cell  *)

Ediskz[z_] =  Edisk[z] [[3]]
(*  quick way to pull out the z component of Edisk[z] *)
k = 1; σ = 1; R = 1;  (* need these defined for the Plot  *)

Plot[Ediskz[z], {z, -10, 10}, PlotRange → {-7, 7},
 AxesLabel → {"z", "Ediskz[z]"}, PlotLabel →
  Style[Framed["E_disk z[z]"], 16, Purple, Bold, Background → Lighter[Green]]]
```

Out[136]= $-\text{If}\left[z<0,\left(1+\frac{z}{\sqrt{R^2+z^2}}\right)(2\,k\,\pi\,\sigma),\ \text{If}\left[z>0,\left(-1+\frac{z}{\sqrt{R^2+z^2}}\right)(2\,k\,\pi\,\sigma)\right]\right]$

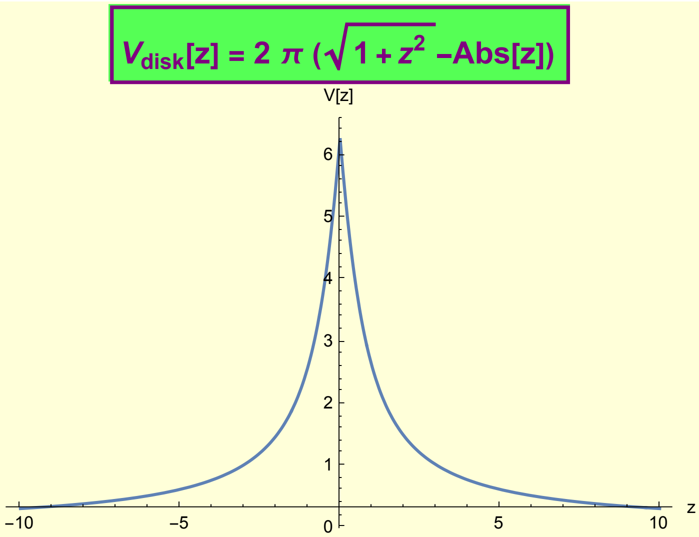
Out[138]=

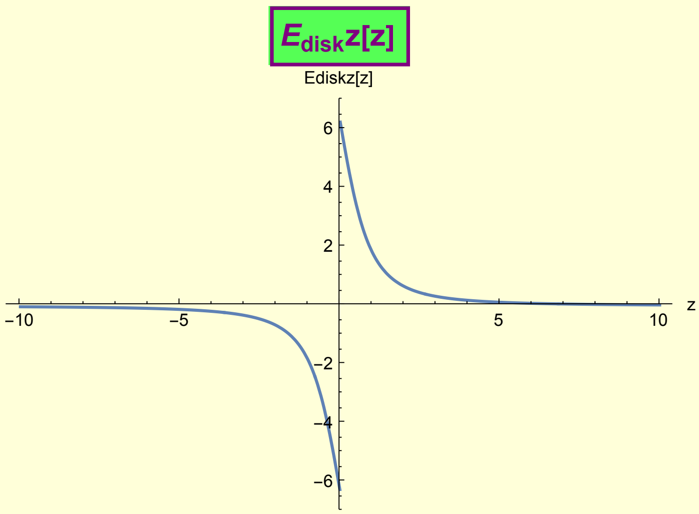


(j) Write down a brief interpretation/discussion about the two plots (e.g., SIGNS and the sign of λ). How are these two plots are related (hint: SLOPE of one of them)??

Enter your Discussion (this is a text cell):

Interpretation: First, we compare this curve with the above Vdisk[*z*], **(I've copied and pasted these plots from above to help you get started:)**





Your Turn:

Note: By looking at $V_{disk}[z]$ we can see where the slopes of $V_{disk}[z]$ are + and –; if we then take the – of these slopes we get the –/+ values of Ediskz[z]. The direction of $\overrightarrow{\text{Edisk}}[z]$ must change because given a $+\sigma$, $\overrightarrow{\text{Edisk}}_z[z]$ will point AWAY from the origin.

This is consistent with the direction of the force on a + test charge q on the two sides of the disk (with $+\sigma$), namely q would be repelled from the disk. Finally, note that the the slope of V is NOT zero at z = 0 (at the center of the ring) and rapidly changes sign.

This implies that Ediskz[z] = z component of – Grad[V] should be non zero AND changes sign as we go from –z to +z (at z = 0); this is clearly seen in the plot of Ediskz[z].

(k) No Brainer - Click inside the cell below (or select it by clicking on the bracket to the right) and execute it (Shift-Return); Answer Boxes will appear; Click on the one you think is correct answer for this question:

Question: Off this symmetry (z) axis, I expect V_{disk} and $\vec{E}_{disk}$ to depend on z only. [Live it up! Click both.]

Imagine moving a +q test charge around the disk with uniform + σ at various x,y,z values off the z axis. I think you can see that the off axis solution: $V_{disk}[x, y, z]$ **depends in general on x,y, AND z.**

In[139]:=

```
Button[
 "1  I agree.  Off the symmetry axis, Vdisk and Edisk depend on z only", {Print[
   "  Wrong --The symmetry of the problem is broken: in Cartesian Coordinates,
     we therefore expect x and/or y dependence to creep in. "]}]
Button["2 I disagree; Off the symmetry axis, Vdisk and EEdisk
   generally do not depend on z only ",
 {Print["  Correct -- The symmetry of the problem is broken; in
     Cartesian Coordinates, we therefore expect x and/or y
     dependence to creep in.\n\nIn Spherical Coordinates one
     would expect θ dependence in V and E, but no ϕ dependence."]}]
```

Out[139]= 1 I agree. Off the symmetry axis, V_{disk} and E_{disk} depend on z only

Out[140]= 2 I disagree; Off the symmetry axis, V_{disk} and EE_{disk} generally do not depend on z only

```
Correct -- The symmetry of the problem is broken; in Cartesian
Coordinates, we therefore expect x and/or y dependence to creep in.

In Spherical Coordinates one would expect θ dependence in V and E, but no ϕ dependence.
```

So No. 2 is the correct answer.

alles Gute!