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An optimal control problem for microwave heating☆

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ABSTRACT

In this paper, we study an optimization problem for a microwave/induction heating process. The cost function is defined such that the temperature profile at the final stage has a relative uniform distribution in the field. The control variable is the applied electric field on the boundary. We show that there exists an optimal electric field which minimizes the cost function. Moreover, a necessary condition for a special case is also derived.

1. Introduction

In modern industries, a convenient method to generate heat is to use microwave and inductive technique (see [1,2] for examples). This method is quick and clean. Moreover, it saves water in sterilization processes for many applications in medical sciences and industrial processes. However, there are still some challenging problems to be resolved in order to improve the current technology. One of these problems is that the heat produced by microwaves is not uniformly distributed in the targeted materials. This causes serious problems in sterilization processes ([3]; also see [4]). There are several reasons for the non-uniformity of the temperature profile. One of those is that various physical parameters such as electric permittivity, electric conductivity, and magnetic permeability may depend on the temperature. Another reason is that the materials may have different structures which affect the heat conductivity.

To improve the uniformity of the temperature profile, one option is to select a suitable applied electric field. In this paper, we formulate this model as an optimal control problem in which the underlying dynamics is governed by Maxwell's equations coupled with nonlinear heat conduction with a source generated by microwaves. The goal of the control is to reach a relatively uniform heat profile at a specific time by controlling the applied field on the boundary of the domain.

Optimal control and controllability problems associated with partial differential equations or systems as underlying states have been studied by many researchers. There are several classical books such as [5,6], and more recent ones such as [7–10]. For control problems associated with Maxwell's equations, there are several papers dealing with the controllability from the boundary (see, for example, [11–17]). When a underlying equation or system is nonlinear, the controllability question generally becomes much more challenging (see [18]). Instead, one often considers optimal control problems by

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selecting a control from the source or from the boundary. The literature on this subject is huge, and there is no way for us to review them here. One fact is clear: when the underlying system is nonlinear, it is important to select a reasonably large admissible set such that the optimal control problem has a solution. We mention [19] by Raymond and Zidani as an example for a general semilinear parabolic equation and [20] by Fu, Yong, and Zhang as an example for a semilinear hyperbolic equation (also see Bardos, Lebeau and Rauch [21]). On the other hand, since the underlying system considered in this paper contains both types of equation, there are only a few results for optimal control problems associated with nonlinear hyperbolic equations due to the complicated well-posedness question for nonlinear wave equations or systems [22]. In this paper, we consider an optimal control problem associated with nonlinear Maxwell's equations coupled with a nonlinear parabolic equation (see the model in Section 2 for details). One of the difficulties in the present paper is that Maxwell's equations are degenerate due to the nonlinearity of the coefficients. Moreover, the nonlinear heat source often belongs to L^1 for this type of coupled systems. This causes a serious problem when one needs to prove the existence of an optimal control. By applying some recent estimates obtained in [23,24] and techniques for elliptic–parabolic equations [22,25], we are able to obtain a better estimate for the nonlinear heat source (see Lemma 3.1 below). With this new estimate, we can overcome the difficulties to establish the existence result. Finally, we would like to mention two papers which are closed related to the present work. In [3], the authors studied a similar problem by selecting a control variable in the heat source. In [4], the authors studied a food sterilization process by using conventional heating method, where the control variable is selected from the boundary. Our present results generalize these previous ones obtained in [4,3].

This paper is organized as follows. In Section 2, for the reader's convenience, we use a unified approach to derive the mathematical model for microwave and induction heating. The optimal control problem is formulated, in which the control function is the applied electric field on the boundary. In Section 3, we prove that the underlying system has a unique weak solution for any applied electric field in a suitable admissible set. Moreover, some important estimates are derived in this section. In Section 4, we show that there exists an optimal control. In Section 5, we derive the necessary condition for the optimal control solution for a special case.

2. The formulation of an optimal control problem

It is well known that there is a fundamental difference between microwave heating and inductive heating [2]. In this paper, we follow the idea from [1,2] by using a unified approach to derive the mathematical model. The advantage of this method is that our model is suitable both for microwave heating and induction heating. The difference between the two heating methods is characterized by physical parameters.

For the reader's convenience, we recall Maxwell's equations and derive the mathematical model. A simplified version has already appeared in [3,23]. Suppose that a targeted substance, say, food-like material, is placed in a microwave processor cavity, denoted by $\Omega \subset R^3$ with C^1-boundary $S = \partial\Omega$. Let $\mathbf{E}(x, t)$ and $\mathbf{H}(x, t)$ denote the electric and magnetic fields at $x \in \Omega$ and time t. Hereafter, a bold letter represents a vector function in R^3. From electromagnetic theory [26], Maxwell's equations in Ω can be expressed by

$$\varepsilon\mathbf{E}_t + \sigma\mathbf{E} = \nabla \times \mathbf{H},$$
$$\mu\mathbf{H}_t + \nabla \times \mathbf{E} = 0,$$
$$\nabla \cdot \mathbf{H} = 0,$$

where Ohm's law $\mathbf{J} = \sigma\mathbf{E}$ is used; ε, μ, and σ are the electric permittivity, magnetic permeability, and electric conductivity, respectively.

We note that $\nabla \cdot \mathbf{H} = 0$ holds automatically as long as an initial field $\mathbf{H}(x, 0)$ satisfies the same condition. Due to the high frequency of the microwaves, the scale of the time variable in electromagnetic fields is very different from that for heat conduction. It is common in practice to assume that the electric and magnetic fields are time harmonic, with fixed frequency ω. With this assumption, Maxwell's equations can be reduced to a Helmholz type of system. Indeed, let

$$\mathbf{E}(x, t) = \hat{\mathbf{E}}(x)e^{i\omega t}, \qquad \mathbf{H}(x, t) = \hat{\mathbf{H}}(x)e^{i\omega t},$$

where i denotes the unit complex number.

From now on, we denote by $\mathbf{E}$ and $\mathbf{H}$ the time-harmonic electric and magnetic fields. Then Maxwell's equations reduce to the following single system for electric field $\mathbf{E}$ (or a similar system for $\mathbf{H}$):

$$\nabla \times [\gamma\nabla \times \mathbf{E}] + \xi\mathbf{E} = 0,$$

where $\gamma = \frac{1}{\mu}$ and $\xi = \omega(-\omega\varepsilon + i\sigma)$.

To model the heat energy produced by microwaves or inductive waves, one has to take into account the dissipative effect. We follow the method in [1] to assume that the electric permittivity ε is characterized by a complex function:

$$\varepsilon = \varepsilon_0(\varepsilon' - i\varepsilon''),$$

where ε_0 is the permittivity in free space, ε' the relative electric permittivity, and ε'' the effective loss factor of electric energy.

Moreover, the magnetic permeability μ is also assumed to be

$$\mu = \mu_0(\mu' - i\mu''),$$

where μ_0 is the permeability in free space, μ' represents the relative magnetic permeability, and μ'' represents the relative magnetic loss factor.

Experiments show that the dielectric coefficients ε' and ε'' strongly depend on the medium and the system temperature, denoted by $u = u(x, t)$ [1,2]. It follows that $\mathbf{E}$ satisfies the following system:

$$\nabla \times [\gamma(x)\nabla \times \mathbf{E}] + \xi(x, u)\mathbf{E} = 0, \quad x \in \Omega,$$

where

$$\xi(x, u) = -a_1(x, u) + ia_2(x, u) := \omega[-\omega\varepsilon_0\varepsilon'(x, u) + i(\sigma(x, u) + \omega\varepsilon_0\varepsilon''(x, u))]$$
$$\gamma(x) = \mu_0\left[\frac{\mu'}{(\mu')^2 + (\mu'')^2} + i\frac{\mu''}{(\mu')^2 + (\mu'')^2}\right].$$

To derive the equation of heat conduction, we need to derive the heat density generated by microwaves. There are two types of current: displacement currents $J_d = \varepsilon(i\omega)\mathbf{E}$ and eddy currents, $J_e = \sigma\mathbf{E}$, by Ohm's law. For a dielectric material, the displacement current dominates the current flow, while for a metallic material, the eddy current dominates the flow. We combine the two types of material by using a unified quantity, denoted by $\mathbf{J}_{total}$. The total current density can be expressed by

$$\mathbf{J}_{total} = \sigma\mathbf{E} + \varepsilon_0(\varepsilon' - i\varepsilon'')i\omega\mathbf{E} = \frac{1}{\omega}[a_2(x, u) - ia_1(x, u)]\mathbf{E}.$$

For microwave heating and inductive heating, the time-average power dissipated in a material per unit volume is given by [1, Chapter 3]

$$Q(x, t) = \frac{1}{2}Re[\mathbf{E}\cdot\mathbf{J}^*_{total}] = \frac{1}{2\omega}a_2(x, u)|\mathbf{E}|^2,$$

where $\mathbf{J}^*_{total}$ represents the complex conjugate of $\mathbf{J}_{total}$ and $a_2(x, u) = \sigma(x, u) + \omega\varepsilon_0\varepsilon''(x, u)$.

With the above local heat source, by using Fourier's law and the conservation of energy, one can easily see that the temperature $u(x, t)$ satisfies a nonlinear heat equation with an internal source generated by microwaves:

$$\rho c u_t - \nabla[k(x, u)\nabla u] = \frac{1}{2\omega}a_2(x, u)|\mathbf{E}|^2, \quad (x, t) \in Q_T,$$

where $Q_T = \Omega \times (0, T]$, ρ is the density, c the specific heat, and $k(x, u)$ the heat conductivity.

We sum up the above derivation and normalize certain physical constants to obtain the following mathematical model:

$$\nabla \times [\gamma(x)\nabla \times \mathbf{E}] + [-a_1(x, u) + ia_2(x, u)]\mathbf{E} = 0, \quad (x, t) \in Q_T, \tag{2.1}$$

$$u_t - \nabla[k(x, u)\nabla u] = \frac{1}{2}a_2(x, u)|\mathbf{E}|^2, \quad (x, t) \in Q_T, \tag{2.2}$$

$$\mathbf{n} \times \mathbf{E} = \mathbf{n} \times \mathbf{G}(x), \quad (x, t) \in S_T = \partial\Omega \times (0, T], \tag{2.3}$$

$$u_{\mathbf{n}}(x, t) = 0, \quad (x, t) \in S_T \tag{2.4}$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \tag{2.5}$$

where $\mathbf{n}$ is the outward unit normal on $S = \partial\Omega$, $u_{\mathbf{n}} = \nabla u \cdot \mathbf{n}$ is the normal derivative on S, and $\mathbf{G}(x)$ is the time-harmonic electric field generated by external optoelectronic devices which is considered as a control variable.

Optimal control problem (P): Given $T > 0$ and a desired temperature $u_T(x) \in L^2(\Omega)$ at time T, find an optimal control $\mathbf{G}^0 \in U_{ad}$ such that the cost functional,

$$J(\mathbf{G}; \mathbf{E}, u) := \int_\Omega |u(x, T) - u_T(x)|^2 dx + \frac{\lambda}{2}\int_{\partial\Omega} |\mathbf{G}(x)|^2 ds, \tag{2.6}$$

reaches its minimum at $(u^0, \mathbf{E}^0)$ for all $\mathbf{G} \in U_{ad}$, where $(\mathbf{E}, u)$ and $(\mathbf{E}^0, u^0)$ are weak solutions of the coupled system (2.1)–(2.5) corresponding to $\mathbf{G}$ and $\mathbf{G}^0$, respectively. U_{ad} is an admissible control set to be specified below. The number $\lambda > 0$ is a typical regularization parameter.

Remark 2.1. In the above model derivation, the electric field $\mathbf{E}$ is assumed to be time harmonic. However, it still depends on time variable from the heat conduction, since the time scale for electric waves is much faster than the time variable in the heat conduction.

3. Estimates of solution for the underlying system

Throughout this paper, Ω is always assumed to be a bounded and simply connected domain in R^3, and the boundary of Ω is C^1-continuous.

We recall some standard Banach spaces. For convenience, a product space B^n is often simply denoted by B. Let

$$H(curl,\Omega)=\{\mathbf{G}(x)\in L^2(\Omega):\nabla\times\mathbf{G}\in L^2(\Omega)\},$$
$$H_0(curl,\Omega)=\{\mathbf{G}(x)\in L^2(\Omega):\nabla\times\mathbf{G}\in L^2(\Omega),\mathbf{n}\times\mathbf{G}=0\ \text{on}\ \partial\Omega\}.$$
$$H(div,\Omega)=\{\mathbf{G}(x)\in L^2(\Omega):\nabla\cdot\mathbf{G}\in L^2(\Omega)\}.$$

$H_0(curl,\Omega)$ and $H(curl,\Omega)$ are Hilbert spaces equipped with inner product

$$\langle\mathbf{G},\mathbf{F}\rangle=\int_\Omega[(\nabla\times\mathbf{G})\times(\nabla\times\mathbf{F}^*)+\mathbf{G}\cdot\mathbf{F}^*]dx,$$

where $\mathbf{F}^*$ represents the complex conjugate of $\mathbf{F}$. $H^1(\Omega)$ is the usual Sobolev space (see [25]).

The admissible control set is

$$U_{ad}=\{\mathbf{G}\in L^2(S):\|\mathbf{G}\|_{L^2(S)}\le A_0<\infty\},$$

where A_0 is a constant.

We impose some basic assumptions which ensure the well-posedness of the underlying system.

H(1) Assume that $u_0(x)$ and $u_T(x)\in L^2(\Omega)$ and nonnegative. The function $k(x,u)$ is measurable in x, uniformly Lipschitz continuous with respect to u, and $0<k_0\le k(x,u)\le k_1$, for positive constants k_0 and k_1.

H(2) (a) Assume that the functions $a_1(x,u)$ and $a_2(x,u)$ are real, measurable, and bounded functions and uniformly Lipschitz continuous with respect to the u-variable. Moreover,
$$0<a_0\le a_1(x,u),a_2(x,u)\le b_0$$
for some constants $a_0>0$, $b_0>0$.

(b) The function $\gamma(x)=\frac{1}{\mu(x)}:=\gamma_1(x)+i\gamma_2(x)$ is assumed to be a bounded complex function, and $\gamma_1(x),\gamma_2(x)\ge\gamma_0>0$ for some constant $\gamma_0>0$.

(c) The boundary function $\mathbf{G}(x)$ is defined on S with an extension such that $\mathbf{G}(x)\in H(curl,\Omega)$ with extended function $\bar{\mathbf{G}}(x)$ satisfies
$$\|\bar{\mathbf{G}}\|_{H(curl,\Omega)}\le c_0\|\mathbf{G}\|_{L^2(S)},$$
where c_0 is a constant that depends only on Ω.

For convenience, we shall denote the extended function $\bar{\mathbf{G}}(x)$ by $\mathbf{G}(x)$ with $\mathbf{G}(x)\in H(curl,\Omega)$.

Set

$$W[0,T]=\{u\in L^2(0,T;H^1(\Omega)),\quad u_t\in L^2(0,T;H^{-1}(\Omega))\}$$

with the norm

$$\|u\|^2_{W[0,T]}=\|u\|^2_{L^2(0,T;H^1(\Omega))}+\|u_t\|^2_{L^2(0,T;H^{-1}(\Omega))}.$$

Then $W[0,T]$ is a reflexive and separable Banach space. The embedding $W[0,T]\hookrightarrow C([0,T];L^2(\Omega))$ is continuous and the mapping from $W[0,T]\hookrightarrow L^2(0,T;L^2(\Omega))$ is compact (see [27]).

Due to the hypotheses H(1)–H(2), for any given $\mathbf{G}(x)\in U_{ad}$, the controlled system (2.1)–(2.5) has a unique weak solution, which we state as follows. The proof can be found in Theorem 4.3 of [24].

Theorem 3.1. *Under the assumptions* H(1) *and* H(2), *the coupled system* (2.1)–(2.5) *has one unique weak solution* $(\mathbf{E},u)$ *with*

$$\mathbf{E}(x,t)-\mathbf{G}(x)\in L^\infty(0,T;H_0(curl,\Omega))\quad\text{and}\quad u(x,t)\in W[0,T].$$

In order to prove the existence of a minimum for the cost functional $J(\mathbf{G};\mathbf{E},u)$, the following estimates for solutions of problem (2.1)–(2.5) are essential for the proof of the main result.

Lemma 3.1. *There are constants $C_1>0$, C_2, and $C_3>0$ which depend on known data such that*

$$\int_\Omega|\nabla\times\mathbf{E}|^2dx+\int_\Omega|\mathbf{E}|^6dx\le C_1\|\mathbf{G}\|^2_{L^2(S)},\tag{3.1}$$

$$\int_\Omega[|\nabla\cdot(a_1\mathbf{E})|^2+|\nabla(a_2\mathbf{E})|^2]dx\le C_2,\tag{3.2}$$

$$\int_\Omega|u|^2dx+\int_0^T\int_\Omega|\nabla u|^2dxdt\le C_3,\tag{3.3}$$

where C_1, C_2, and C_3 are constants which depend only on known data.

Proof. The main idea of the proof follows from [24].

Step 1: Derivation for the first part of estimate (3.1).

We recall by assumption H(2) that $\mathbf{G}(x)$ can be extended to Ω with

$$\|\mathbf{G}\|_{H(curl,\Omega)} \le c_0\|\mathbf{G}\|_{L^2(S)}.$$

To derive the estimate, we take the inner product by $\mathbf{E}^* - \mathbf{G}^*$ with system (2.1) to obtain

$$\begin{aligned}&\int_\Omega \gamma(x)|\nabla\times\mathbf{E}|^2dx + \int_\Omega [-a_1(x,u)+ia_2(x,u)]|\mathbf{E}|^2dx\\ &\quad= \int_\Omega \gamma(x)(\nabla\times\mathbf{E})\cdot(\nabla\times\mathbf{G}^*)dx + \int_\Omega [-a_1(x,u)+ia_2(x,u)]\mathbf{E}\cdot\mathbf{G}^*dx\\ &\quad:= I_1+I_2,\end{aligned}$$

where $\mathbf{E}^*$ and $\mathbf{G}^*$ represent the complex conjugate of $\mathbf{E}$ and $\mathbf{G}$, respectively.

Taking the imaginary part first, and then the real part from the above identity, and using assumption H(2), we have

$$\gamma_0\int_\Omega |\nabla\times\mathbf{E}|^2dx \le b_0\int_\Omega |\mathbf{E}|^2dx + |Re(I_1+I_2)|. \tag{3.4}$$

$$\gamma_0\int_\Omega |\nabla\times\mathbf{E}|^2dx + a_0\int_\Omega |\mathbf{E}|^2dx \le |Im(I_1+I_2)|. \tag{3.5}$$

It follows from the Cauchy–Schwarz inequality with small parameters $\delta_1 > 0$ and $\delta_2 > 0$ that

$$|I_1| \le \delta_1\int_\Omega |\nabla\times\mathbf{E}|^2dx + C(\delta_1)\int_\Omega |\nabla\times\mathbf{G}|^2dx$$
$$|I_2| \le \delta_2\int_\Omega |\mathbf{E}|^2dx + C(\delta_2)\int_\Omega |\mathbf{G}|^2dx.$$

From (3.5), we choose δ_1 sufficiently small to obtain

$$(a_0-\delta_1)\int_\Omega |\mathbf{E}|^2dx \le \delta_2\int_\Omega |\nabla\times\mathbf{E}|^2dx + C(\delta_1,\delta_2)\int_\Omega [|\mathbf{G}|^2 + |\nabla\times\mathbf{G}|^2]dx. \tag{3.6}$$

Substitute (3.6) to (3.4) and choose δ_2 sufficiently small to get

$$\int_\Omega |\nabla\times\mathbf{E}|^2dx \le C\left(\int_\Omega |\nabla\times\mathbf{G}|^2dx + \int_\Omega |\mathbf{G}|^2dx\right).$$

It follows from the Extension Theorem [22, p. 254] that

$$\int_\Omega |\nabla\times\mathbf{E}|^2dx \le C\left(\int_\Omega [|\nabla\times\mathbf{G}|^2 + |\mathbf{G}|^2]dx\right) \le C_1\|\mathbf{G}\|^2_{L^2(S)},$$

where $C_1 > 0$ depends only on known data in H(1)–H(2) and Ω.

Step 2: Derivation of the estimate (3.2).

We may assume that $\mathbf{E}$ is smooth with respect to spatial variables. Otherwise, we just use smooth approximation to derive the estimate and then take the limit. Now, by taking divergence directly to Eq. (2.1),

$$\nabla\{\nabla\times(\gamma(x)\nabla\times\mathbf{E}) + (-a_1(x,u)+ia_2(x,u))\mathbf{E}\} = 0,$$

we use the fact the $div(curl\mathbf{F}) = 0$ to obtain

$$\nabla\cdot\big([-a_1(x,u)+a_2(x,u)]\mathbf{E}\big) = 0, \quad x\in\Omega.$$

It follows that

$$\int_\Omega |\{\nabla\cdot([-a_1(x,u)+ia_2(x,u)]\mathbf{E})\}|^2dx = 0.$$

Consequently, there exists a constant C_2 such that

$$\int_\Omega [|\nabla\cdot(a_1\mathbf{E})|^2 + |\nabla\cdot(a_2\mathbf{E})|^2]dx \le C_2.$$

Step 3: Derivation for the second part of the estimate (3.1).

Since $a_1 \geq a_0 > 0$, we can now use the decomposition property [28] to see that $\mathbf{E}(x) \in H^1(\Omega)$ and

$$\|\mathbf{E}\|_{H^1(\Omega)} \leq C\left[\int_\Omega |\mathbf{E}|^2|\nabla\times\mathbf{E}|^2 + |\nabla(a_1\mathbf{E})|^2dx + \int_S |\mathbf{G}|^2ds\right].$$

By applying Sobolev's embedding with the dimension $N = 3$, we see that

$$\|\mathbf{E}\|_{L^6(\Omega)} \leq C + C\int_S |\mathbf{G}|^2ds,$$

where C depends only on C_1 and C_2 in Steps 1 and 2 and the domain Ω.

Step 4: Derivation of the estimate (3.3).

Since the right-hand side of Eq. (2.2) is in $L^\infty(0, T; L^3(\Omega))$, by using the standard energy estimate, we see that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^2dx + \int_\Omega |\nabla u|^2dx &\leq C\int_\Omega ua_2(x, u)|\mathbf{E}|^2dx\\ &\leq C\int_\Omega u^2dx + C\int_\Omega |\mathbf{E}|^4dx,\end{aligned}$$

where C depends only on the constants in H(1)–H(2) and those in the estimates (1) and (2). Finally, Gronwall's inequality yields the desired estimate (3.3). □

4. Existence of an optimal control

In this section, we will prove the existence of an optimal control for problem (P). We define a feasible control set as follows.

$$\begin{aligned}A_{ad} = \{&(\mathbf{G}, \mathbf{E}, u) \in L^2(S) \times H(curl, \Omega) \times W[0, T] :\\ &(\mathbf{E}, u) \text{ is a solution of system (2.1)–(2.5) corresponding to } \mathbf{G} \in U_{ad}\}.\end{aligned}$$

Theorem 4.1. *Suppose that the assumptions* H(1) *and* H(2) *hold. Then there exists an optimal control for problem* (P).

Proof. From Theorem 3.1, we know that, for every $\mathbf{G} \in U_{ad}$, there exists a unique solution $(\mathbf{E}, u)$ to system (2.1)–(2.5). Assume that

$$\inf_{\mathbf{G}\in U_{ad}} J(\mathbf{G}; \mathbf{E}, u) = J^* < +\infty.$$

Let $\{\mathbf{G}_m\} \subset U_{ad}$ be a minimizing sequence such that

$$\lim_{m\to+\infty} J(\mathbf{G}_m; \mathbf{E}_m, u_m) = J^*,$$

where $(\mathbf{E}_m, u_m)$ is the solution of system (2.1)–(2.5) corresponding to the control $\mathbf{G}_m$ for $m = 1, 2, \ldots$. That is,

$$\nabla\times[\gamma(x)\nabla\times\mathbf{E}_m] + [-a_1(x, u_m) + ia_2(x, u_m)]\mathbf{E}_m = 0, \quad (x, t) \in Q_T, \tag{4.1}$$

$$(u_m)_t - \nabla[k(x, u_m)\nabla u_m] = \frac{1}{2}a_2(x, u_m)|\mathbf{E}_m|^2, \quad (x, t) \in Q_T, \tag{4.2}$$

$$\mathbf{n}\times\mathbf{E}_m(x) = \mathbf{n}\times\mathbf{G}_m(x), \quad x \in S_T, \tag{4.3}$$

$$(u_m)_{\mathbf{n}}(x, t) = 0, \quad (x, t) \in S_T, \tag{4.4}$$

$$u_m(x, 0) = u_0(x), \quad x \in \Omega. \tag{4.5}$$

It follows from the estimates in Lemma 3.1 and the weak compactness of $H(curl, \Omega)$ that there exists a subsequence of $\{\mathbf{G}_m(x)\}$, again denoted by $\{\mathbf{G}_m(x)\}$, such that

$$\mathbf{G}_m(x) \longrightarrow \mathbf{G}^0(x) \quad \text{weakly in } H(curl, \Omega)$$

as $m \to +\infty$. Moreover, by the Mazur lemma, we know that $\mathbf{G}^0(x) \in U_{ad}$.

From Lemma 3.1, we know that $\{\mathbf{E}_m\}$ is bounded in $L^\infty(0, T; H(curl, \Omega)) \bigcap L^6(Q_T)$, $\{u_m\}$ is bounded in $W[0, T]$, and $\max_{0\leq t\leq T}\|u_m(t)\|_{L^2(\Omega)} \leq C$ for some constant $C > 0$. Noting that the embedding operators from $H(curl, div_a, \Omega) \hookrightarrow L^2(\Omega)$

and from $W[0,T] \hookrightarrow L^2(0,T;L^2(\Omega))$ are compact, we conclude that there exists a subsequence of $\{\mathbf{E}_m, u_m\}$, relabelled as $\{\mathbf{E}_m\, u_m\}$ again, and $\mathbf{E}^0 \in H(curl,\Omega)\bigcap L^\infty(0,T;L^6(\Omega))$, $u^0 \in W[0,T]$, such that

$$\mathbf{E}_m \longrightarrow \mathbf{E}^0 \quad \text{weakly in } H(curl,\Omega) \tag{4.6}$$

$$\nabla u_m \longrightarrow \nabla u^0 \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)) \tag{4.7}$$

$$\mathbf{E}_m \longrightarrow \mathbf{E}^0 \quad \text{strongly in } L^2(\Omega) \tag{4.8}$$

$$u_m \longrightarrow u^0 \quad \text{strongly in } L^2(0,T;L^2(\Omega)) \tag{4.9}$$

$$u_m(x,t) \longrightarrow u^0(x,t) \quad \text{weakly in } L^2(\Omega), \text{ for any } t \in [0,T] \tag{4.10}$$

as $m \to +\infty$.

From the strong convergence of L^2-space, we can further select a subsequence such that

$$\mathbf{E}_m(x,t) \longrightarrow \mathbf{E}^0(x,t) \quad \text{a.e. } (x,t) \in Q_T, \tag{4.11}$$

$$u_m(x,t) \longrightarrow u^0(x,t) \text{ for any } t \in [0,T], \text{ a.e. } x \in \Omega. \tag{4.12}$$

Note that

$$\begin{aligned}
&\int_0^T \int_\Omega |[a_2(x,u_m)|\mathbf{E}_m|^2 - a_2(x,u^0)|\mathbf{E}^0|^2]|^2 dxdt \\
&\quad \le \int_0^T \int_\Omega |[a_2(|\mathbf{E}_m|^2 - |\mathbf{E}^0|^2)]|^2 dxdt + \int_0^T \int_\Omega |[(a_2(x,u_m) - a_2(x,u^0))|\mathbf{E}^0|^2]|^2 dxdt \\
&\quad \le A_0^2 \int_0^T \int_\Omega |a_2(x,u_m)|^2 (|\mathbf{E}_m|^2 - |\mathbf{E}^0|^2)^2 dxdt + \int_0^T \int_\Omega |[(a_2(x,u_m) - a_2(x,u^0))|\mathbf{E}^0|^2]|^2 dxdt,
\end{aligned}$$

where b_0 is the upper bound of $a_2(x,u)$.

Since a_2 is bounded, by (4.5), (4.6), and the bound of $\mathbf{E}^m$ in $L_6(\Omega)$, we obtain by using the dominance convergence theorem that

$$\int_0^T \int_\Omega |(|\mathbf{E}_m|^2 - |\mathbf{E}^0|^2)|^2 dxdt \longrightarrow 0$$

and

$$\int_0^T \int_\Omega |(a_2(x,u_m) - a_2(x,u^0))|\mathbf{E}^0|^2|^2 dxdt \longrightarrow 0,$$

as $m \to +\infty$. Hence,

$$\int_0^T \int_\Omega |a_2(x,u_m)|\mathbf{E}_m|^2 - a_2(x,u^0)|\mathbf{E}^0|^2|^2 dxdt \longrightarrow 0$$

as $m \to +\infty$. That is,

$$a_2(x,u_m)|\mathbf{E}_m|^2 \longrightarrow a_2(x,u^0)|\mathbf{E}|^2, \quad \text{strongly in } L^2(0,T;L_2(\Omega)). \tag{4.13}$$

Multiplying Eq. (4.2) by $u_m(x,t) - u^0(x,t)$ and then taking integration over Q_T, we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega (u_m)_t \cdot (u_m - u^0) dxdt - \int_0^T \int_\Omega \nabla(k(u_m,x)\nabla u_m) \cdot (u_m - u^0) dxdt \\
&\quad = \int_0^T \int_\Omega \frac{1}{2} a_2(x,u_m)|\mathbf{E}_m|^2 \cdot (u_m - u^0) dxdt.
\end{aligned}$$

By using integration by parts, then taking the limit and using (4.9) and (4.13), one can see that

$$\lim_{m\to+\infty} \int_0^T \int_\Omega k(x,u_m)|\nabla u_m|^2 = \int_0^T \int_\Omega k(x,u^0)|\nabla u^0|^2 dxdt.$$

One concludes that u^0 satisfies the following equation in a weak sense:

$$u_t^0 - \nabla(k(x,u^0)\nabla u^0) = \frac{1}{2} a_2(x,u^0)|\mathbf{E}^0|^2.$$

By performing integration by parts and taking the limit for all test functions

$$\psi \in C^\infty(0,T), \qquad \xi \in H^1(\Omega),$$

we have

$$\lim_{m+\to}\left[\int_\Omega u_m(x,T)\psi(T)\xi dx - \int_\Omega u_m(x,0)\psi(0)\xi dx\right] = \int_\Omega u^0(x,T)\psi(T)\xi dx - \int_\Omega u^0(x,0)\psi(0)\xi dx.$$

Taking $\psi(T)=0$ and $\psi(0)=1$, we obtain

$$\int_\Omega u_0(x)\xi dx = \int_\Omega u^0(x,0)\xi dx$$

for arbitrary $\xi \in H^1(\Omega)$. Hence

$$u^0(x,0)=u_0(x) \quad \text{in } L^2(\Omega).$$

Now, $\forall \mathbf{\Phi} \in H_0(curl, \Omega)$,

$$\int_\Omega (\gamma(x)\nabla\times\mathbf{E}_m)\cdot(\nabla\times\mathbf{\Phi})dx + \int_\Omega(-a_1(x,u_m)+ia_2(x,u_m))\mathbf{E}_m\cdot\mathbf{\Phi}dx = 0.$$

That is,

$$\begin{aligned}&\int_\Omega (\gamma(x)\nabla\times\mathbf{E}_m)\cdot(\nabla\times\mathbf{\Phi})dx + \int_\Omega(-a_1(x,u)+ia_2(x,u))\mathbf{E}_m\cdot\mathbf{\Phi}dx\\ &\quad = \int_\Omega[(a_1(x,u)-a_1(x,u_m)+i(a_2(x,u_m)-a_2(x,u)))]\mathbf{E}_m\cdot\mathbf{\Phi}dx. \qquad (4.14)\end{aligned}$$

By the uniformly Lipschitz condition in H(2), one can easily see that

$$\begin{aligned}&\left|\int_\Omega (a_1(x,u^0)-a_1(x,u_m)+i(a_2(x,u^0)-a_2(x,u_m)))\mathbf{E}_m\cdot\mathbf{\Phi}dx\right|\\ &\quad\le\left(\int_\Omega|(a_1(x,u^0)-a_1(x,u_m)+i(a_2(x,u_m)-a_2(x,u^0)))|^2dx\right)^{1/2}\cdot\left(\int_\Omega|\mathbf{E}_m|^2|\mathbf{\Phi}|^2dx\right)^{1/2}\\ &\quad\le L\left|\left(\int_\Omega u_m(x,t)-u^0(x,t)\,|^2\,dx\right)^{1/2}\left(\int_\Omega|\mathbf{E}_m|^2|\mathbf{\Phi}|^2dx\right)^{1/2}\right|\\ &\quad\longrightarrow 0,\quad \text{as } m\to\infty,\end{aligned}$$

where $L>0$ is a constant which depends on the Lipschitz constants for $a_1(x,u^0)$ and $a_2(x,u^0)$.

By taking $m\to+\infty$ in (4.14) and using (4.8), we obtain that $\mathbf{E}^0$ is a weak solution of (2.1)–(2.5) corresponding to $\mathbf{G}^0$. Therefore

$$(\mathbf{G}^0,\mathbf{E}^0,u^0)\in A_{ad}$$

and

$$\begin{aligned}J^* &= \lim_{m\to+\infty} J(\mathbf{G}_m,\mathbf{E}_m,u_m)\\ &= \lim_{m\to+\infty}\left[\int_\Omega|u_m(x,T)-u_T(x)|^2dx+\frac{\lambda}{2}\int_S|\mathbf{G}_m|^2dx\right]\\ &\ge \int_\Omega|u^0(x,T)-u_T(x)|^2dx+\frac{\lambda}{2}\int_S|\mathbf{G}^0|^2dx\\ &= J(\mathbf{G}^0,\mathbf{E}^0,u^0)\ge J^*.\end{aligned}$$

This shows that J attains its minimum at $(\mathbf{G}^0;\mathbf{E}^0,u^0)\in A_{ad}$. □

5. Necessary conditions for an optimal solution

In deriving the necessary conditions, one often needs additional regularity for the weak solution of system (2.1)–(2.5). For technical reasons we have to focus on a special case by assuming that the electric field $\mathbf{E}(x,t)$ lies in one direction, i.e. $\mathbf{E}(x,t)=\{0,0,w(x,t)\}$ with $x=(x_1,x_2)\in R^2$. For this case, we can derive more regularity which is needed in the derivation of the necessary conditions. The general case relies on the regularity of the weak solution, which is quite complicated, and will be carried out in a separate paper.

With this assumption, the Maxwell system in Section 2 can be written as a scalar elliptic equation for $w(x,t)$. Then the underlying system becomes the following coupled elliptic–parabolic system (see [24]):

$$-\nabla[\gamma(x)\nabla w]+(-a_1(x,u)+ia_2(x,u))w=0,\quad (x,t)\in Q_T, \tag{5.1}$$

$$u_t-\nabla[k(x,u)\nabla u]=\frac{1}{2}a_2(x,u)|w|^2,\quad (x,t)\in Q_T, \tag{5.2}$$

$$w(x,t)=g(x),\quad (x,t)\in S_T, \tag{5.3}$$

$$u_{\mathbf{n}}(x,t)=0,\quad (x,t)\in S_T, \tag{5.4}$$

$$u(x,0)=u_0(x),\quad x\in\Omega. \tag{5.5}$$

We recall the optimal control problem for the current case. Given $T>0$ and a desired temperature $u_T(\cdot)\in L^2(\Omega)$ at $t=T$, find an optimal control $g^0(x)\in U_{ad}$ associated with solution (w^0,u^0) of (5.1)–(5.5) such that

$$J(g^0;w^0,u^0)\le J(g;w,u):=\frac{1}{2}\int_\Omega|u(x,T)-u_T(x)|^2dx+\frac{\lambda}{2}\int_{\partial\Omega}|g|^2ds \tag{5.6}$$

for all $g\in U_{ad}$, where (w,u) is a solution to the coupled system (5.1)–(5.5). We also need to modify the admissible set to allow more regularity for the solution. Let the admissible control set $U_{ad}=\{g\in L^\infty(S):\|g\|_{L^\infty(S)}\le g_0\}$.

H(5.1) (a) Assume that $\gamma(x)$ is of class $C^\alpha(\Omega)$. There exist positive constants γ_0 and γ_1 such that
$0<\gamma_0\le Re(\gamma(x)),\ Im(\gamma(x))\le\gamma_1<\infty$.
(b) The function $k(x,u)$ is Hölder continuous with respect to x and Lipschitz continuous with respect to the u-variable. Moreover, there exist two positive constants k_0 and k_1 such that
$0<k_0\le k(x,u)\le k_1<\infty$.
(c) The functions $\xi(x,u):=-a_1(x,u)+ia_2(x,u)\in C^{\alpha,1+\alpha}(\Omega\times R)$, and there exists a constant A_0 such that
$|\xi(x,u)|\le A_0$.

H(5.2) $u_0(x)\in C^\alpha(\bar\Omega)$ and $u_0(x)\ge 0$.

Theorem 5.1. *Under assumptions* H(5.1)–H(5.2), *problem* (5.1)–(5.5) *has a unique weak solution* $(w,u)\in H^1(\Omega)\bigcap L^\infty(Q_T)\times L^2(0,T;H^1(\Omega))C^{\alpha,\frac{\alpha}{2}}(\bar Q_T)$ *in* Q_T *for any* $T>0$. *Moreover,*

$$\|w\|_{H^1(\Omega)}+\|w\|_{L^\infty(\Omega)}\le C_1,$$
$$\|u\|_{C^{\alpha,\frac{\alpha}{2}}(\bar Q_T)}+\|u\|_{L^2(0,T;H^1(\Omega))}\le C_2,$$

where C_1 and C_2 depend only on known data.

Proof. The proof follows the same steps as that for Theorem 3.1 (also see Theorem 5.1 of [24]). Here we only show that $w(x,t)$ is uniformly bounded and that u is Hölder continuous in $\bar Q_T$. Let

$$M=\sup_{x\in S}|g(x)|.$$

We define

$$w^+(x,t)=Re(w^+)+iIm(w^+):=(w-M)^+,$$

where

$$Re(w^+):=(Re(w)-M)^+,\qquad Im(w^+):=(Im(w)-M)^+.$$

By taking the real part of Eq. (5.1) and then multiplying by $Re(w^+)$, we have

$$\int_\Omega\gamma(x)|\nabla w^+|^2dx+\int_\Omega[-a_1(x,u)]Im(w)Re(w^+)dx=0.$$

Similarly, we have a similar equality after taking the imaginary part of Eq. (5.1).

Since $a_1(x,u)$ and $a_2(x,u)$ are bounded and $Re(\gamma(x))\ge\gamma_0$, we see that

$$\int_\Omega|\nabla w^+|^2dx\le C\int_\Omega|w^+|^2dx.$$

By using standard Moser iteration, we see that $w^+=0$ in Ω for all $t\in[0,T]$. It follows that

$$\sup_{\{0\le t\le T\}}\|w\|_{L^\infty(\Omega)}\le M.$$

Now, since w is bounded, the right-hand side of Eq. (5.2) is uniformly bounded. Since $k(x,u)$ is bounded with a positive lower bound, it follows that $u(x,t)$ is Hölder continuous in $\bar Q_T$ by the classical theory for parabolic equations, and the estimate for $u(x,t)$ holds (see [25]). □

To obtain the necessary conditions for the optimal control problem, we differentiate the objective functional with respect to the control variable. Since the objective functional depends on u, which is coupled with w through the coupled system (5.1)–(5.2), we have to differentiate u and w with respect to control g.

Theorem 5.2. *Under assumptions* H(5.1)–H(5.2), *the mapping* $g \mapsto (w, u)$ *is differentiable in the following sense:*

$$W_\varepsilon(x,t) := \frac{w(g+\varepsilon h) - w(g)}{\varepsilon} \longrightarrow W, \quad \text{weakly-* in } L^\infty(0,T;H^1(\Omega)), \tag{5.7}$$

$$U_\varepsilon(x,t) := \frac{u(g+\varepsilon h) - u(g)}{\varepsilon} \longrightarrow U(x,t), \quad \text{weakly in } W[0,T], \tag{5.8}$$

for any $g, h \in U_{ad}$ *such that* $g + \varepsilon h \in U_{ad}$ *for small* ε. $W \in W_0[0,T]$ *and* $U \in H^1(0,T;H(curl,\Omega))$ *satisfy*

$$U_t - \nabla[k(x,u)\nabla U] = \nabla[(k_u(x,u)\nabla u)U] + \frac{1}{2}(a_2)_u(x,u)|w|^2 U + a_2(x,u)wW, \quad \text{in } Q_T, \tag{5.9}$$

$$\nabla[\gamma(x)\nabla W] + [-a_1(x,u) + ia_2(x,u)]W = [(a_1)_u(x,u) - i(a_2)_u(x,u)]wU, \quad \text{in } Q_T, \tag{5.10}$$

$$U_{\mathbf{n}}(x,t) = 0, \quad \text{in } S_T \tag{5.11}$$

$$W(x,t) = h(x), \quad \text{in } S_T, \tag{5.12}$$

$$U(x,0) = 0 \quad \text{in } \Omega, \tag{5.13}$$

where we denote by $u = u(g)$ *and* $w = w(g)$ *the solution of* (5.1)–(5.5) *corresponding to* $g(x)$.

Proof. Now we also denote by $u^\varepsilon = u(g^\varepsilon)$, $w^\varepsilon = w(g^\varepsilon)$ the solution of (5.1)–(5.5) corresponding to $g^\varepsilon := g + \varepsilon h$ for any $h \in U_{ad}$. We present the proof in the following steps.

Step 1: There exist constants C_1 and C_2 such that

$$\int_\Omega [|\nabla W_\varepsilon|^2 + |W_\varepsilon|^2]dx \le C_1,$$

$$\sup_{0\le t\le T} \int_\Omega |U_\varepsilon|^2 dx + \int\int_{Q_T} |\nabla U_\varepsilon|^2 dxdt \le C_2.$$

Indeed, since $(w^\varepsilon, u^\varepsilon)$ and (w, u) are solutions of system (5.1)–(5.5) corresponding to g^ε and g, respectively, we have

$$-\nabla[\gamma(x)\nabla W_\varepsilon] + (-a_1(x,u^\varepsilon) + ia_2(x,u^\varepsilon))W_\varepsilon = \frac{1}{\varepsilon}[(-a_1(x,u^\varepsilon) + a_1(x,u)) + i(a_2(x,u^\varepsilon) - a_2(x,u))]w, \quad (x,t) \in Q_T, \tag{5.14}$$

$$U_{\varepsilon t} - \nabla[k(x,u^\varepsilon)\nabla U_\varepsilon] = \nabla\left[\left(\frac{k(x,u^\varepsilon) - k(x,u)}{\varepsilon}\right)\nabla u\right] + \frac{1}{2\varepsilon}a_2(x,u^\varepsilon)|w^\varepsilon|^2 - \frac{1}{2\varepsilon}a_2(x,u)|w|^2, \quad (x,t) \in Q_T, \tag{5.15}$$

$$W_\varepsilon(x,t) = h(x), \quad (x,t) \in S_T, \tag{5.16}$$

$$U_{\varepsilon\mathbf{n}}(x,t) = 0, \quad (x,t) \in S_T \tag{5.17}$$

$$U_\varepsilon(x,0) = 0, \quad x \in \Omega. \tag{5.18}$$

Multiplying Eq. (5.14) by $(W_\varepsilon - h)^*$ and integrating over Ω, we find that

$$\int_\Omega \gamma(x)|\nabla W_\varepsilon|^2 dx + \int_\Omega [-a_1(x,u^\varepsilon) + ia_2(x,u^\varepsilon)]|W_\varepsilon|^2 dx = I_1 + I_2 + I_3,$$

where

$$I_1 := \int_\Omega \gamma(x)(\nabla W_\varepsilon)^* \cdot (\nabla h)dx;$$

$$I_2 := \int_\Omega [-a_1(x,u^\varepsilon) + ia_2(x,u^\varepsilon)]W_\varepsilon \cdot h^* dx;$$

$$I_3 := \int_\Omega \frac{1}{\varepsilon}[(-a_1(x,u^\varepsilon) + a_1(x,u)) + i(a_2(x,u^\varepsilon) - a_2(x,u))]w \cdot (W_\varepsilon - h)^* dx.$$

Using the same energy method as in Lemma 3.1, we obtain

$$\int_\Omega |\nabla W_\varepsilon|^2 dx + \int_\Omega |W_\varepsilon|^2 dx \le C + C\int_\Omega \frac{|a_1(x,u^\varepsilon) - a_1(x,u)|^2 + |a_2(x,u^\varepsilon) - a_2(x,u)|^2}{\varepsilon^2}|w|^2 dx.$$

By the Lipschitz condition for a_1 and a_2 in assumption H(5.2) and the fact that $w \in L^\infty(Q_T)$,

$$\int_\Omega [|\nabla W_\varepsilon|^2 dx + |W_\varepsilon|^2] dx \le C + C_1 \int_\Omega |U_\varepsilon|^2 dx. \tag{5.19}$$

Similarly, multiplying Eq. (5.15) by U_ε and integrating over Ω, after some routine calculation we obtain

$$\frac{d}{dt}\int_\Omega |U_\varepsilon|^2 dx + \int_\Omega |\nabla U_\varepsilon|^2 dx \le C_2 \int_\Omega |U_\varepsilon|^2 dx + C_3 \int_\Omega |W_\varepsilon|^2 dx \tag{5.20}$$

for some constants $C_2 > 0$ and $C_3 > 0$ which depend only on known data, but not on ε.

We use the estimate (5.19) in (5.20) to obtain

$$\frac{d}{dt}\int_\Omega |U_\varepsilon|^2 dx + \int_\Omega |\nabla U_\varepsilon|^2 dx \le C + C_2 \int_\Omega |U_\varepsilon|^2 dx.$$

Gronwall's inequality implies that

$$\sup_{0\le t\le T} \|U_\varepsilon\|^2_{L_2(\Omega)} \le C \tag{5.21}$$

for some constant $C > 0$ which depends only on known data, but not on ε.

Now we use the estimate (5.22) in (5.19) to obtain

$$\max_{0\le t\le T} \|U_\varepsilon\|^2_{L_2(\Omega)} + \int_0^T \int_\Omega |\nabla U_\varepsilon||^2 dxdt \le C_4, \tag{5.22}$$

$$\int_\Omega [|\nabla W_\varepsilon|^2 + |W_\varepsilon|^2] dx \le C_5 \tag{5.23}$$

for some constants $C_4 > 0$ and $C_5 > 0$ which depend only on known data, but not on ε.

Step 2. Now we are ready to derive system (5.9)–(5.12). The estimates (5.23) and (5.24) imply that there exists a subsequence of $\varepsilon \to 0$ and there exist $U \in W[0, T]$ and $W \in H^1(\Omega)$ such that

$$\begin{aligned}
&U_\varepsilon = \frac{u^\varepsilon - u}{\varepsilon} \longrightarrow U, \quad \text{weakly in } W[0, T];\\
&W_\varepsilon = \frac{w^\varepsilon - w}{\varepsilon} \longrightarrow W, \quad \text{weakly in } H^1(\Omega);\\
&U_\varepsilon \longrightarrow U, \quad \text{strongly in } L_2(Q_T);\\
&W_\varepsilon \longrightarrow W, \quad \text{strongly in } L^\infty(0, T; L_2(\Omega))
\end{aligned}$$

as $\varepsilon \to 0$. Furthermore, by selecting a subsequence if necessary, for a.e. $t \in [0, T]$,

$$U_\varepsilon \longrightarrow U(x, t), \quad \text{a.e., } x \in \Omega; \tag{5.24}$$

$$W_\varepsilon \longrightarrow W(x, t), \quad \text{a.e., } x \in \Omega \tag{5.25}$$

as $\varepsilon \to 0$.

By definitions of weak solutions for system (5.1)–(5.5) (see [24]) and (5.14), (5.15), we have

$$\begin{aligned}
&\int_\Omega \gamma(x)\nabla W_\varepsilon \nabla v dx + \int_\Omega (-a_1(x, u^\varepsilon) + ia_2(x, u^\varepsilon))W_\varepsilon v dx\\
&\quad = \frac{1}{\varepsilon}\int_\Omega [-a_1(x, u^\varepsilon) + a_1(x, u) + i(a_2(x, u^\varepsilon) - a_2(x, u))] wv dx,
\end{aligned} \tag{5.26}$$

for all $v \in H_0^1(\Omega)$, and

$$\begin{aligned}
-\int_0^T \int_\Omega U_\varepsilon \phi_t dxdt + \int_0^T \int_\Omega k(x, u^\varepsilon)\nabla U_\varepsilon \cdot \nabla\phi dxdt = &-\int_0^T \int_\Omega \left[\frac{k(x, u^\varepsilon) - k(x, u)}{\varepsilon}\right] \nabla u \cdot \nabla\phi dxdt\\
&+ \frac{1}{2\varepsilon}\int_0^T \int_\Omega (a_2(x, u^\varepsilon)|w^\varepsilon|^2 - a_2(x, u)|w|^2)\phi dxdt,
\end{aligned} \tag{5.27}$$

for all $\phi \in W[0, T]$ with $\phi(x, T) = 0$.

On the one hand, the terms on the left-hand side of (5.26) can be shown to converge to the limit function by weak compactness. On the other hand, we use the mean-value theorem to see that each term in the right-hand side of (5.26) can be written as follows:

$$\int_\Omega \left\{ \frac{-a_1(x,u^\varepsilon)+a_1(x,u)}{\varepsilon} \right\} wvdx \to \int_\Omega (a_1)_u(x,u)Uvwdx;$$
$$\int_\Omega \left\{ \frac{a_2(x,u^\varepsilon)-a_2(x,u)}{\varepsilon} \right\} wdx \to \int_\Omega (a_2)_u(x,u)Uvwdx.$$

The terms on the left-hand side of (5.27) can be shown to converge by weak compactness too. The terms on the right-hand side of (5.27) can be shown similarly.

$$\int_0^T \int_\Omega \frac{k(x,u^\varepsilon)-k(x,u)}{\varepsilon} \nabla u \nabla \phi dx \longrightarrow \int_0^T \int_\Omega k_u(x,u)U\nabla u \cdot \nabla \phi dx.$$
$$\frac{1}{2\varepsilon}\int_0^T \int_\Omega (a_2(x,u^\varepsilon)|w^\varepsilon|^2 - a_2(x,u)|w|^2)\phi dxdt$$
$$= \frac{1}{2\varepsilon}\int_0^T \int_\Omega (a_2(x,u^\varepsilon)(|w^\varepsilon|^2 - |w|^2)\phi dxdt + \frac{1}{2\varepsilon}\int_0^T \int_\Omega (a_2(x,u^\varepsilon) - a_2(x,u))|w|^2)\phi dxdt$$
$$\to \frac{1}{2}\int_0^T \int_\Omega (a_2(x,u)_u)U|w|^2\phi dxdt$$

as $\varepsilon \to 0$. That is, $U(x,t)$ and $W(x,t)$ satisfy (5.9) and (5.10). The rest of the initial and boundary conditions can be shown by choosing suitable test functions by the classical method. □

Theorem 5.3. *Suppose that assumptions* H(5.1)–H(5.2) *hold. If $g^0 \in U_{ad}$ is an optimal control with the corresponding states (w^0,u^0), then there exist $(p,q) \in W[0,T] \times H(0,T;H^1(\Omega))$ such that*

$$-\nabla[\gamma(x)\nabla w^0] + (-a_1^0 + ia_2^0)w^0 = 0, \quad (x,t) \in Q_T, \tag{5.28}$$
$$u_t^0 - \nabla[k(x,u^0)\nabla u^0] = \frac{1}{2}a_2^0|w^0|^2, \quad (x,t) \in Q_T, \tag{5.29}$$
$$w^0(x,t) = g^0(x), \quad (x,t) \in S_T, \tag{5.30}$$
$$u_{\mathbf{n}}^0(x,t) = 0, \quad (x,t) \in S_T, \tag{5.31}$$
$$u^0(x,0) = u_0(x), \quad x \in \Omega, \tag{5.32}$$

the adjoint system

$$p_t - \nabla[k(x,u)\nabla p] = \frac{1}{2}(a_2^0)_u|w^0|^2 p + (-(a_1^0)_u + i(a_2^0)_u)w^0 q, \quad \text{in } Q_T, \tag{5.33}$$
$$\nabla(\gamma(x)\nabla q) + [(-a_1^0 + ia_2^0)q - a_2^0]w^0 p = 0, \quad \text{in } Q_T, \tag{5.34}$$
$$p_{\mathbf{n}}(x,t) = 0, \quad (x,t) \in S_T \tag{5.35}$$
$$q(x,t) = 0, \quad (x,t) \in S_T, \tag{5.36}$$
$$p(x,T) = u^0(x,T) - u_T(x), \quad x \in \Omega, \tag{5.37}$$

where $a_1^0 = a_1(x,u^0)$ and $a_2^0 = a_2(x,u^0)$. The following inequality is satisfied.

$$\int_0^T \int_S \gamma(x)q + \lambda g^0(g - g^0)dsdt \geq 0, \quad \forall g \in U_{ad}. \tag{5.38}$$

Proof. Similar to system (5.1)–(5.5), since w^0, $(a_1^0)_u$, (a_2^0) are bounded, one can easily show that there exists a unique solution for the adjoint system. The rest of the theorem is similar to the standard calculation for a single elliptic or parabolic equation (see [10]). We skip the details here. □

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